A NOTE ON MULTIPLICATIVELY PRIME IDEALS OF OPERATORS

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ABSTRACT. An ideal $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$ is said to be multiplicatively prime if $AXB \in \mathcal{I}$ for all $X \in \mathcal{L}(\mathcal{H})$ implies A or B is in $\mathcal{I}$. In this paper, we show there exists a nonnorm multiplicatively prime ideal other than $\mathcal{F}$.

An ideal $\mathcal{I}$ of $\mathcal{L}(\mathcal{H})$ is said to be *square* (respectively, *multiplicatively prime*) if $\mathcal{I} = \mathcal{I}^2$ (respectively, $AXB \in \mathcal{I}$ for all $X \in \mathcal{L}(\mathcal{H})$ implies A or B in $\mathcal{I}$). An ideal $\mathcal{I}$ is said to have the square root property if $\sqrt{|A|} \in \mathcal{I}$ for all $A \in \mathcal{I}$. R. I. Loebl [L] proved that if $\mathcal{I}$ is multiplicatively prime then $\mathcal{I}$ is square and $\mathcal{I}$ has the square root property. He also showed that these properties are equivalent if $\mathcal{I}$ is a proper norm ideal. (In this article, norm ideals are named "symmetrically normed Banach ideals" [G-K, p. 68]. A symmetrically normed ideal is said to be a nonnorm ideal if it is not a Banach space under any symmetrical norm.) In this case, $\mathcal{I}$ is either $\{0\}$ or $\mathcal{K}$, the ideal of all compact operators. He asked whether there exists a nonnorm multiplicatively prime ideal other than $\mathcal{F}$, the ideal of all finite rank operators. The purpose of this paper is to show there exists such an ideal.

Let $\{a_i\}, \{b_i\}$ be two decreasing sequences whose limits are zero. Then $\{a_i\} \oplus \{b_i\} = \{c_i\}$ denotes the decreasing sequence which is the union of $\{a_i\}$ and $\{b_i\}$ (counting multiplicity). For any natural number n , $n \odot \{a_i\}$ denotes the sequence

$$\{a_i\} \oplus \{a_i\} \oplus \overset{n \text{ times}}{\dots} \oplus \{a_i\}.$$

Recall that an *ideal set* I is a collection of decreasing sequences $\{a_i\}$ with the following properties:

- (i) If $\{a_n\} \in I$, then $\lim_{n \rightarrow \infty} a_n = 0$.
- (ii) If $\{a_n\} \in I$ and $\{b_n\}$ is a decreasing sequence such that $0 \leq b_n \leq a_n$ for all n , then $\{b_n\} \in I$.
- (iii) If $\{a_n\} \in I$ and $c > 0$ then $\{ca_n\} \in I$.
- (iv) If $\{a_n\}, \{b_n\} \in I$, then $\{a_n\} \oplus \{b_n\} \in I$.

An ideal set I is said to have the square root property if $\{a_n\} \in I$ implies $\{a_n^{1/2}\} \in I$. An ideal set I is said to be multiplicatively prime if for any two decreasing sequences $\{a_n\}, \{b_n\}, \{a_nb_n\} \in I$ implies $\{a_n\} \in I$ or $\{b_n\} \in I$. It is

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known [C and F-L] that there is a bijection between (multiplicatively prime) proper two-sided ideals and (multiplicatively prime) ideal sets. Hence, we only need to construct a nontrivial multiplicatively prime ideal set. We need a couple of lemmas.

LEMMA 1. Suppose the ideal set I has the square root property and $\{a_n\}$ is a decreasing sequence which is not in I . Let $\{b_n\}$ and $\{c_n\}$ be two decreasing sequences such that $\{b_n c_n\} \in I$. Then there exists an ideal set $I' \supseteq I$ with the square root property so that either $\{b_n\}$ or $\{c_n\}$ is in I' but I' does not contain $\{a_n\}$.

LEMMA 2. Suppose the ideal set I has the square root property and $\{a_n\}$ is a decreasing sequence which is not in I . Then there exists a multiplicatively prime ideal set $I' \supseteq I$ such that $\{a_n\} \notin I'$.

PROOF OF LEMMA 2. Suppose Lemma 1 were proved. Let $\mathcal{D}$ be the collection of the ideal set $I' \supseteq I$ with the square root property such that $\{a_n\} \notin I'$. If I' and I'' are in $\mathcal{D}$, we say $I' \leq I''$ whenever $I' \subseteq I''$. Then $(\mathcal{D}, \leq)$ satisfies the assumption of Zorn's Lemma, and $\mathcal{D}$ contains a maximal element I' . By Lemma 1, I' is multiplicatively prime. $\square$

By Lemma 2 and Theorem 3 in [L], we have the following theorem.

THEOREM. $\mathcal{F}$ is the unique nonnorm multiplicatively prime ideal if (and only if) $\mathcal{F}$ is the unique nonnorm ideal with the square root property.

REMARK 1. Since

$$\mathcal{J}_p = \bigcup_{k \geq 0} C_{p^{2^k}} \neq \mathcal{F}$$

is a nonnorm ideal with the square root property [L], there exists a nonnorm multiplicatively prime ideal other than $\mathcal{F}$.

REMARK 2. Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} b_n$. We say $\{a_n\} \leq \{b_n\}$ if there exists N such that $a_n \leq b_n$ whenever $n > N$. If $c \geq c' \geq 0$, $m \geq m' \geq 1$, $k \geq k' \geq 0$, and $\{a_n\} \leq \{b_n\}$, then

$$m' \odot \{c'a_n^{1/2^{k'}}\} \leq m' \odot \{c'a_n^{1/2^k}\} \leq m \odot \{ca_n^{1/2^k}\} \leq m \odot \{cb_n^{1/2^k}\}.$$

If I is an ideal set, $\{b_n\} \in I$, and $\{a_n\} \leq \{b_n\}$, then by (ii) and (iii) $\{a_n\} \in I$.

REMARK 3. Suppose I is an ideal set with the square root property. Let $\{a_n\}$ be a decreasing sequence such that $\lim_{n \rightarrow \infty} a_n = 0$. Then the smallest ideal set with the square root property which contains I and $\{a_n\}$ consists of all sequences $\{b_n\}$ with the following property: There exist $c \geq 0$, $m \geq 1$, $k \geq 0$, and $\{c_n\} \in I$ such that

$$\{b_n\} \leq m \odot \{ca_n^{1/2^k}\} \oplus \{c_n\}.$$

PROOF OF LEMMA 1. Suppose Lemma 1 is not true. Then there exist an ideal set I with the square root property and three decreasing sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ such that $\{a_n\} \notin I$, $\{b_n\} \notin I$, $\{c_n\} \notin I$, $\{a_n b_n\} \in I$, and the smallest ideal set I_1 (respectively I_2) which contains I and $\{a_n\}$ (respectively $\{b_n\}$) contains $\{c_n\}$. For fixed $c > 0$, $m > 0$, and $k \geq 0$, if $\{d_n\} = m \odot \{ca_n^{1/2^k}\}$ and $\{d'_n\} = m \odot \{cb_n^{1/2^k}\}$ then $\{d_n d'_n\} \in I$. By Remarks 2 and 3, we may assume that there exist $\{e_n\}$,

$\{e'_n\} \in I$ such that

$$\{c_n\} \leq \{a_n\} \oplus \{e_n\} = \{f_n\}$$

(respectively, $\{c_n\} \leq \{b_n\} \oplus \{e'_n\} = \{f'_n\}$). One can verify that $\{f_n f'_n\} \leq \{a_n b_n\} \oplus \{e_n\} \oplus \{e'_n\} \in I$. This implies $\{c_n^2\} \in I$. But I has the square root property. So $\{c_n\} \in I$ and we get a contradiction. $\square$

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