STABILITY AND SKEW-PRODUCT FLOWS

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ABSTRACT. We consider a skew-product flow with a minimal flow as the basic flow, and we show that, if a bounded solution is positively uniformly stable, then the ω -limit set of it is a minimal set. This is an extension of a result in [2], where it was assumed that the basic flow is equicontinuous.

Let R denote the set of real numbers and R^n the Euclidean n -space with the usual norm $|\cdot|$. Let X be a metric space with the metric d_X . A continuous mapping $\pi: X \times R \rightarrow X$ is called a *flow on (a phase space) X* if π satisfies the following two conditions:

- (1) $\pi(x, 0) = x$ for $x \in X$.
- (2) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for $x \in X$ and $t, s \in R$.

The orbit of π through $x \in X$ is denoted by $C_\pi(x) = \{\pi(x, t); t \in R\}$. The closure of a set $A \subset X$ is denoted by $\bar{A}$. The ω -limit set of $x \in X$ is denoted by $\Omega^+(x) = \{y \in X; \pi(x, t_n) \rightarrow y \text{ for some sequence } \{t_n\} \subset R \text{ such that } t_n \rightarrow \infty \text{ as } n \rightarrow \infty\}$. A nonempty set $M \subset X$ is called an *invariant set of π* if we have $C_\pi(x) \subset M$ for every $x \in M$. The restriction of π to an invariant set M is denoted by $\pi|_M$. A nonempty compact set $M \subset X$ is called a *minimal set of π* if we have $\overline{C_\pi(x)} = M$ for each $x \in M$. If X is itself a minimal set of π , we say that π is a *minimal flow on X* . Let π_1 and π_2 be flows on X_1 and X_2 , respectively. A continuous mapping h from X_1 onto X_2 is called a *homomorphism from π_1 to π_2* if we have $h(\pi_1(x_1, t)) = \pi_2(h(x_1), t)$ for $(x_1, t) \in X_1 \times R$. A homomorphism h from π_1 to π_2 is said to be of *negatively distal type* if for each $x_2 \in X_2$ and for a pair of distinct points $x'_1, x''_1 \in h^{-1}(x_2)$ we have

$$\inf_{t \leq 0} \{d_{X_1}(\pi_1(x'_1, t), \pi_1(x''_1, t))\} > 0.$$

If further $\inf_{t \in R} \{d_{X_1}(\pi_1(x'_1, t), \pi_1(x''_1, t))\} > 0$, we say that h is of *distal type*. Let σ be a flow on X . We consider a skew-product flow $\pi = (\sigma, \phi)$ on $X \times R^n$, that is,

$$\pi((x, y), t) = (\sigma(x, t), \phi(x, y, t))$$

for $(x, y) \in X \times R^n$ and $t \in R$. $\pi = (\sigma, \phi)$ become a flow on $X \times R^n$ if and only if ϕ satisfies the following two conditions.

- (1) $\phi(x, y, 0) = y$ for $(x, y) \in X \times R^n$.
- (2) $\phi(x, y, t + s) = \phi(\sigma(x, t), \phi(x, y, t), s)$ for $(x, y) \in X \times R^n$ and $t, s \in R$.

We say that σ is the *basic flow of π* , and that $\phi(x, y, t)$ is the *solution through (x, y)* .

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A solution $\phi(x, y, t)$ through (x, y) is said to be *positively uniformly stable* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\phi(x, y, \tau + t) - \phi(\sigma(x, \tau), y', t)| < \varepsilon \quad (\text{for } t \geq 0),$$

whenever $\tau \in R$ and $|\phi(x, y, \tau) - y'| < \delta$.

LEMMA 1. Let σ be a minimal flow on a compact metric space X , and $\pi = (\sigma, \phi)$ a skew-product flow on $X \times R^n$. If a bounded solution $\phi(x, y, t)$ through $(x, y) \in X \times R^n$ is positively uniformly stable, then for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for each $(x^*, y^*) \in \Omega^+(x, y)$ we have

$$|\phi(x^*, y^*, \tau + t) - \phi(\sigma(x^*, \tau), y', t)| < \varepsilon \quad (\text{for } t \geq 0),$$

whenever $\tau \in R$ and $|\phi(x^*, y^*, \tau) - y'| < \delta$. We say that the function $\delta(\varepsilon)$ is the modulus of uniform stability for $\phi(x, y, t)$.

PROOF. See Theorem 4 in [2].

Let X be a metric space. The set of compact subsets of X is denoted by $\text{Com}(X)$. For $K \in \text{Com}(X)$, and for $r > 0$, we denote the set $\{x \in X; d_X(x, K) < r\}$ by $V_r(K)$. For $K_1, K_2 \in \text{Com}(X)$ we define

$$\rho(K_1, K_2) \in \inf\{r; K_2 \subset V_r(K_1) \text{ and } K_1 \subset V_r(K_2)\}.$$

Then ρ is a metric on $\text{Com}(X)$. By this ρ $\text{Com}(X)$ becomes a metric space.

LEMMA 2. Let X and Z be metric spaces, and let h be a mapping from Z into $\text{Com}(X)$. Define a mapping k from $X \times Z$ to R by $k(x, z) = d_X(x, h(z))$ for $(x, z) \in X \times Z$. If h is continuous, then k is also continuous.

PROOF. Easy.

THEOREM. Let σ be a minimal flow on a compact metric space X , and let $\pi = (\sigma, \phi)$ be a skew-product flow on $X \times R^n$. If a bounded solution $\phi(x, y, t)$ through $(x, y) \in X \times R^n$ is positively uniformly stable, then $\Omega^+(x, y)$ is a compact minimal set of π .

PROOF. Since X is compact and $\phi(x, y, t)$ is bounded for $t \geq 0$, $\Omega^+(x, y)$ is compact. Let $M \subset \Omega^+(x, y)$ be a minimal set of π . We shall prove that $M = \Omega^+(x, y)$. We consider the following metric on $X \times R^n$:

$$d_{X \times R^n}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + |y_1 - y_2|$$

for $(x_1, y_1), (x_2, y_2) \in X \times R^n$. The restriction $P|M$ of the projection $P: X \times R^n \rightarrow X$ to M is a homomorphism from $\pi|M$ to σ , and it is of negatively distal type [2, p. 19]. Hence it is of distal type [1, p. 102]. Consequently $h: X \rightarrow \text{Com}(M)$ defined by $h(x) = M \cap P^{-1}(x)$ for $x \in X$ is continuous [1, p. 105]. Since $\text{Com}(M) \subset \text{Com}(X \times R^n)$, h is continuous as the mapping from X to $\text{Com}(X \times R^n)$. Put $h(x) = \{x\} \times M_x$ ($M_x \subset R^n$), $U_r(x) = \{y \in R^n; d_{R^n}(y, M_x) < r\}$ for $r > 0$, and $U_r = \bigcup_{x \in X} \{x\} \times U_r(x)$. Then U_r is an open subset of $X \times R^n$ which contains M . In fact, if U_r is not open, then there exist $(x_0, y_0) \in U_r$ and $(x_n, y_n) \notin U_r$ ($n = 1, 2, \dots$) such that $(x_n, y_n) \rightarrow (x_0, y_0)$ as $n \rightarrow \infty$. Since $(x_n, y_n) \notin U_r$, we have $d_{R^n}(y_n, M_{x_n}) \geq r$ for $n = 1, 2, \dots$. Since

$$d_{X \times R^n}((x_0, y_0), h(x_0)) = d_{X \times R^n}((x_0, y_0), x_0 \times M_{x_0}) = d_{R^n}(y_0, M_{x_0}),$$

we have

$$\lim_{n \rightarrow \infty} d_{X \times R^n}((x_n, y_n), h(x_n)) = d_{X \times R^n}((x_0, y_0), h(x_0)) = d_{R^n}(y_0, M_{x_0}) \geq r$$

by Lemma 2. This implies that $(x_0, y_0) \notin U_r$. This is a contradiction. Hence U_r is open in $X \times R^n$.

Further, it is easy to verify that $\{U_r\}_{r>0}$ is a fundamental system of neighborhoods of M . If $\Omega^+(x, y)$ is not minimal, then there exists $(x_0, y_0) \in \Omega^+(x, y) - M$. Next choose a neighborhood V of M such that $(x_0, y_0) \notin \bar{V}$. Then there exists an $\varepsilon > 0$ such that $U_\varepsilon \subset V$. Take a point $(x', y') \in U_\varepsilon$, where $\delta(\varepsilon) > 0$ is the modulus of uniform stability for the solution $\phi(x, y, t)$. Choose $(x', y^*) \in M$ such that $|y' - y^*| < \delta$. Then, by Lemma 1, we have $|\phi(x', y^*, t) - \phi(x', y', t)| < \varepsilon$ for $t \geq 0$. Since M is an invariant set of π , we have $(\sigma(x', t), \phi(x', y^*, t)) \in M$ for $t \geq 0$, and hence $\phi(x', y', t) \in U_\varepsilon(\sigma(x', t))$ for $t \geq 0$. This implies that $(\sigma(x', t), \phi(x', y', t)) = \pi((x', y'), t) \in U_\varepsilon$ for $t \geq 0$. Since $M \subset \Omega^+(x, y)$ and U_δ is an open neighborhood of M , there exists $\tau > 0$ such that $\pi((x, y), \tau) \in U_\delta$. Hence, by the above, $\pi(\pi((x, y), \tau), t) = \pi((x, y), \tau + t) \in U_\varepsilon$ for $t \geq 0$. This implies that $\Omega^+(x, y) \subset \bar{U}_\varepsilon \subset \bar{V}$. Since $(x_0, y_0) \in \Omega^+(x, y)$ and $(x_0, y_0) \notin \bar{V}$, this is a contradiction. Hence we can conclude $\Omega^+(x, y) = M$, and hence $\Omega^+(x, y)$ is a minimal set of π .

REMARK 1. In [2] R. J. Sacker and G. R. Sell showed that if σ is equicontinuous in the Theorem, then $\Omega^+(x, y)$ is a distal minimal set of π . The Theorem is an extension of this result.

REMARK 2. Let $C(R, R^n)$ be the set of continuous functions from R to R^n with the compact-open topology. Define a flow η on $C(R, R^n)$ by $\eta(f, t) = f_t$ for $(f, t) \in C(R, R^n) \times R$, where $f_t(s) = f(t + s)$ for $s \in R$. We denote the hull of f , $\{\overline{f_t}\}_{t \in R}$, by $H(f)$. We say that f is a *recurrent function* if $H(f)$ is a compact minimal set of η . We can easily show that for every $(x^*, y^*) \in \Omega^+(x, y)$ in the Theorem, $\phi(x^*, y^*, \cdot)$ is a recurrent function.

REMARK 3. Let W be an open subset of R^n , and let $C(W \times R, R^n)$ be the set of continuous functions from $W \times R$ to R^n with the compact-open topology. Define a flow ξ on $C(W \times R, R^n)$ by $\xi(f, t) = f_t$ for $(f, t) \in C(W \times R, R^n) \times R$, where $f_t(x, s) = f(x, t + s)$ for $(x, s) \in W \times R$. We denote the hull of f , $\{\overline{f_t}\}_{t \in R}$, by $H_\xi(f)$. We say that, if $H_\xi(f)$ is a compact minimal set of ξ , f is a *recurrent function*. Let f be a recurrent function. We assume that for $g \in H_\xi(f)$ and $x_0 \in W$ there is a unique solution $\phi(g, x_0, t)$ of the initial valued problem

$$x' = g(x, t), \quad \phi(g, x_0, 0) = x_0,$$

defined on R . Then $\pi = (\xi|H_\xi(f), \phi)$ is a skew-product flow which satisfies the conditions in the Theorem.

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