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REVERSE HÖLDER INEQUALITIES FOR SPHERICAL HARMONICS

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ABSTRACT. We prove that the L^p-norm with respect to the normalized Lebesgue measure on the sphere of any spherical harmonic of degree k is bounded by a constant independent of the dimension times its L^2-norm. Several consequences are obtained from this result.

1. Introduction. In a previous paper [**2**] we were led to study the quotient $\|Y_k\|_p/\|Y_k\|_2$ where Y_k stands for a spherical harmonic of degree k in $\mathbf{R}^n$ and the L^p-norms are taken with respect to the normalized Lebesgue measure on the sphere S^{n-1}, i.e.,

$$\|Y_k\|_p^p = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} |Y_k(u)|^p \, d\sigma(u)$$

($d\sigma(u)$ =Lebesgue measure on S^{n-1}, $|S^{n-1}|$ =measure of $S^{n-1} = 2\pi^{n/2}\Gamma(n/2)^{-1}$).

When $p > 2$, Hölder's inequality provides the trivial lower bound 1. We prove here, by using two different approaches, that we have the upper bound $(p-1)^{k/2}$, independent of n. The first proof is the same as in [**2**] with further precision; the second uses two well-known but far from trivial facts: the Bochner-Hecke formula and the Beckner-Hausdorff-Young inequality.

For $p < 2$, Hölder's inequality gives the upper bound 1 and a lower bound independent of n can be found by using the preceding part and interpolation.

Similar results can be obtained for any polynomial of degree k and any sphere in $\mathbf{R}^n$ using the decomposition of the polynomial restricted to the sphere as a sum of spherical harmonics. Some other consequences are also given.

We use the same notation for the spherical harmonic Y_k defined on S^{n-1} and the solid harmonic defined in all $\mathbf{R}^n$ which are related by $Y_k(x) = Y_k(x/|x|)|x|^k$.

We are indebted to José L. Rubio de Francia for suggesting certain applications and improvements to this paper.

2. The main theorem and its two proofs.

THEOREM 1. *Let Y_k be a spherical harmonic of degree k in $\mathbf{R}^n$. Then, if $p \geq 2$,*

$$\|Y_k\|_p \leq (p-1)^{k/2}\|Y_k\|_2.$$

PROOF. We use induction on k. Let $k = 1$. By rotation it is enough to prove the theorem for $Y_1(u) = u_1$. But a simple computation gives

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} |u_1|^p \, d\sigma(u) = \frac{\Gamma(n/2)\Gamma((p+1)/2)}{\pi^{1/2}\Gamma((n+p)/2)}$$

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and $\|Y_1\|_2 = n^{-1/2}$; then it follows from the properties of the gamma function that

$$\|Y_1\|_p/\|Y_1\|_2 \leq (p-1)^{1/2}.$$

Assume now the theorem for $k-1$ and let Y_k be a spherical harmonic of degree k. We claim that it is enough to obtain

$$\|Y_k\|_p \leq \left(\frac{p-1}{k(kp+n-2)}\right)^{1/2} \||\nabla Y_k|\|_p \quad \text{with equality for } p=2. \tag{1}$$

In fact, the induction hypothesis and Minkowski's inequality imply

$$\| |\nabla Y_k| \|_p \leq (p-1)^{(k-1)/2} \| |\nabla Y_k| \|_2$$

and then using (1) successively with L^p- and L^2-norms, we get

$$\begin{aligned}\|Y_k\|_p &\leq \left(\frac{p-1}{k(kp+n-2)}\right)^{1/2} (p-1)^{(k-1)/2} \| |\nabla Y_k| \|_2 \\ &= (p-1)^{k/2} \left(\frac{2k+n-2}{kp+n-2}\right)^{1/2} \|Y_k\|_2 \leq (p-1)^{k/2} \|Y_k\|_2.\end{aligned}$$

Finally we prove (1). From the homogeneity of solid harmonic Y_k and Green's formula we have

$$\begin{aligned}\int_{S^{n-1}} |Y_k|^p \, d\sigma &= \frac{1}{k} \int_{S^{n-1}} |Y_k|^{p-1} \operatorname{sgn} Y_k \frac{\partial Y_k}{\partial \nu} \, d\sigma \\ &= \frac{1}{k} \int_{|x|<1} \nabla(|Y_k|^{p-1} \operatorname{sgn} Y_k) \nabla Y_k \, dx \\ &= \frac{p-1}{k} \int_{|x|<1} |Y_k|^{p-2} |\nabla Y_k|^2 \, dx \\ &= \frac{p-1}{k(kp+n-2)} \int_{S^{n-1}} |Y_k|^{p-2} |\nabla Y_k|^2 \, d\sigma.\end{aligned}$$

When $p=2$ we get the equality in (1); for $p>2$, just apply Hölder's inequality with exponents $p/(p-2)$ and $p/2$. □

The second way to prove Theorem 1 gives a somewhat more precise result:

THEOREM 1′. *Let Y_k be a spherical harmonic of degree k, $2 \leq p < \infty$ and $1/p + 1/q = 1$. Then,*

$$\|Y_k\|_p \leq (p-1)^{k/2} \|Y_k\|_q.$$

Theorem 1 follows immediately because of $\|Y_k\|_q \leq \|Y_k\|_2$.

PROOF. Bochner-Hecke's formula states that

$$\mathfrak{F}(e^{-\pi|x|^2} Y_k(x)) = i^{-k} e^{-\pi|\xi|^2} Y_k(\xi) \tag{2}$$

($\mathfrak{F}$ is the Fourier transform and Y_k is here the solid harmonic; see Stein [**4**, p. 71].) The Hausdorff-Young inequality can be written in the form

$$\|\mathfrak{F}(f)\|_{L^p(\mathbf{R}^n)} \leq (q^{1/q}/p^{1/p})^{n/2} \|f\|_{L^q(\mathbf{R}^n)}$$

(see Beckner [**1**]). When this inequality is applied to (2), taking into account that

$$\|e^{-\pi|x|^2} Y_k(x)\|^p_{L^p(\mathbf{R}^n)} = \frac{\Gamma((kp+n)/2)}{\pi^{kp/2} p^{(kp+n)/2} \Gamma(n/2)} \|Y_k\|_p^p$$

we get

$$\frac{\|Y_k\|_p}{\|Y_k\|_q} \le \left(\frac{p}{q}\right)^{k/2} \frac{\Gamma((kq+n)/2)^{1/q}\Gamma(n/2)^{1/p}}{\Gamma((kp+n)/2)^{1/p}\Gamma(n/2)^{1/q}}. \tag{3}$$

Since $\log\Gamma$ is a convex function in $(0,\infty)$, the last factor is ≤ 1 and the theorem is proved. □

The function $Y_k(x) = (x_1 + ix_2)^k$ is harmonic and homogeneous of degree k. If we compute the L^p-norm of its restriction to S^{n-1} we see that the preceding results are sharp in the following sense: for constants independent of n, no exponent less than $k/2$ can appear in the right-hand side of Theorems 1 and 1′; in fact

$$\sup_n \frac{\|Y_k\|_p}{\|Y_k\|_2} \le c(k)p^{k/2}.$$

3. Some consequences. (a) Theorem 1 and the method of rotations give the following: *Let $\{Y_j\}$ be a basis of the linear space of spherical harmonics of degree k in $\mathbf{R}^n$ and $d(k,n)$ its dimension. If we normalize the Y_j in such a way that $\|Y_j\|_2 = d(k,n)^{-1/2}$ and define the operators $(R_jf)\hat{}(\xi) = Y_j(\xi/|\xi|)\hat{f}(\xi)$, there then exists $C_{p,k}$ independent of n such that*

$$\left\| \left(\sum_{j=1}^{d(k,n)} |R_jf|^2 \right)^{1/2} \right\|_p \le C_{p,k}\|f\|_p, \qquad 1 < p < \infty,$$

with $C_{p,k} = O(p^{1+k/2})$, $p \to \infty$, and $= O((p-1)^{-1})$, $p \to 1$. This was our motivation for Theorem 1 and can be seen in [**2**].

(b) For $p < 2$, a lower bound for $\|Y_k\|_p/\|Y_k\|_p$ can be obtained from Theorem 1, namely

COROLLARY 2. *If $0 < p < 2$,*

$$\|Y_k\|_2 \le e^{k((2/p)-1)}\|Y_k\|_p.$$

PROOF. Let $s > 2$. By interpolation and Theorem 1

$$\|Y_k\|_2 \le \|Y_k\|_p^\theta \|Y_k\|_s^{1-\theta} \le (s-1)^{(1-\theta)k/2}\|Y_k\|_p^\theta\|Y_k\|_2^{1-\theta}$$

with $1/2 = \theta/p + (1-\theta)/s$. Then,

$$\|Y_k\|_2 \le (s-1)^{(1-\theta)/\theta\cdot k/2}\|Y_k\|_p$$

and the corollary follows from

$$\inf_{s>2}(s-1)^{(1-\theta)/\theta} = \lim_{s\to 2}(s-1)^{(1-\theta)/\theta} = e^{2(2/p-1)}. \quad \square$$

(c) Theorem 1 and Corollary 2 have the following similar versions in the case of arbitrary polynomials.

COROLLARY 3. *Let P_k be any polynomial of degree k and S any sphere in $\mathbf{R}^n$. Then, if $2 < p < \infty$,*

$$\left(\frac{1}{|S|}\int_S |P_k|^p\,d\sigma\right)^{1/p} \le p^{k/2}\left(\frac{1}{|S|}\int_S |P_k|^2\,d\sigma\right)^{1/2}$$

and if $0 < p < 2$,

$$\left(\frac{1}{|S|}\int_S |P_k|^2\,d\sigma\right)^{1/2} \le 4^{k(2/p-1)}\left(\frac{1}{|S|}\int_S |P_k|^p\,d\sigma\right)^{1/p}.$$

PROOF. Since translation and dilation change a polynomial into another of the same degree, it will be enough to prove the result for $S = S^{n-1}$. But on S^{n-1}, $P_k = \sum_{j=0}^k Y_j$ where Y_j is a spherical harmonic of degree j. By Theorem 1,

$$\|P_k\|_p \le \sum_{j=0}^{k} \|Y_j\|_p \le \sum_{j=0}^{k} (p-1)^{j/2}\|Y_j\|_2$$

and the first part of the corollary is a consequence of the orthogonality of the Y_j after applying Cauchy-Schwarz inequality.

The second part follows from the first as in the proof of Corollary 2. □

(d) The size of the constants in Theorem 1 and Corollary 3 makes it possible to give an estimate of exponential type with constant independent of n.

COROLLARY 4. *Let P_k be a polynomial of degree k in $\mathbf{R}^n$. Then, for any sphere S in $\mathbf{R}^n$*

$$\frac{1}{|S|}\int_S \exp\left|\frac{P_k(u)}{\|P_k\|_2}\right|^{\lambda} d\sigma(u) \le C(k,\lambda)$$

with a constant independent of n if $\lambda < 2/k$ (also for $\lambda = 2/k$ if $k \ge 6$).

(e) The following application is based on a result in probability theory.

Let $Y^{(n)} = (Y_1, \ldots, Y_n, 0, 0, \ldots)$, $n = 1, 2, \ldots$, be random variables such that $(Y_1, \ldots, Y_n)$ are uniformly distributed in a sphere of $\mathbf{R}^n$ of radius $r_n = (n/2\pi)^{1/2}$ and let $X = (X_1, \ldots, X_m, \ldots)$ be a random variable with $X_1, \ldots, X_m, \ldots$ independent and having $\exp(-\pi|x|^2)$ as distribution function. Then, the sequence $Y^{(n)}$ converges to X in law.

This result is known and can be easily proved. If $E_n(f)$ stands for the expectation of f with respect to $Y^{(n)}$ and $E(f)$ is the expectation with respect to X, we have

$$\lim_{n\to\infty} E_n(f) = E(f).$$

Taking now as f the pth power of a polynomial of degree k and using Corollary 3 we have

COROLLARY 5. *Let $X = (X_1, \ldots, X_n, \ldots)$ be a random variable where the X_i are independent and have $\exp(-\pi|x|^2)$ as distribution function. If $P(X)$ is a polynomial of degree k in the variables $X_1, \ldots, X_n, \ldots$ the following reverse Hölder inequalities hold.*

$$E(|P|^p)^{1/p} \le p^{k/2} E(|P|^2)^{1/2}, \qquad 2 < p < \infty,$$
$$E(|P|^2)^{1/2} \le 4^{k(2/p-1)} E(|P|^p)^{1/p}, \qquad 0 < p < 2.$$

(f) Upper bounds for the quotient $\|Y_k\|_p/\|Y_k\|_2$, $2 < p < \infty$, are interesting also in the study of Bochner-Riesz operators on the sphere. In that case the dimension n of the underlying space is kept fixed and the interest is in the behaviour of the quotient for $k \to \infty$. Sharp bounds have been obtained by C. Sogge [**3**].

$$\|Y_k\|_p/\|Y_k\|_2 = O(k^{\alpha(p)}) \tag{4}$$

where

$$\alpha(p) = (n-2)(p-2)/4p \quad \text{if } 2 \le p \le 2n/(n-2),$$
$$= (n-2)/2 - (n-1)/p \quad \text{if } 2n/(n-2) \le p \le \infty.$$

From the proof of Theorem 1′ and more precisely from inequality (3) it follows immediately that

$$\|Y_k\|_p/\|Y_k\|_q = O(k^{(n-1)(p-2)/2p}).$$

But, if we put $\|Y_k\|_2$ instead of $\|Y_k\|_q$ the bound we get for the quotient is not sharp and it does not apply to obtaining nontrivial results for Bochner-Riesz operators.

Using (4) we can get in a trivial way the bound $O(k^{2\alpha(p)})$ for the quotient $\|Y_k\|_p/\|Y_k\|_q$. It is easily verified that $2\alpha(p)$ is less than our exponent when p is close to 2 and bigger when p is close to ∞ (for $n > 3$).

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