SOME NEW MULTIPLIERS OF FOURIER SERIES

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ABSTRACT. Let L^1 be the space of all complex-valued 2π -periodic integrable functions f and let $\widehat{L^1}$ be the space of sequences of Fourier coefficients $\hat{f}$. A sequence λ is an $(\widehat{L^1} \rightarrow \widehat{L^1})$ multiplier if $\lambda \cdot \hat{f} = (\lambda(n)\hat{f}(n))$ belongs to $\widehat{L^1}$ for every f in L^1 . The space of even sequences of bounded variation is defined by $bv = \{\lambda \mid \lambda_n = \lambda_{-n}, \sum_{k=0}^{\infty} |\Delta\lambda_k| + \sup_n |\lambda_n| < \infty\}$, where $\Delta\lambda_k = \lambda_k - \lambda_{k+1}$ and the space of even bounded quasiconvex sequences is defined by $q = \{\lambda \mid \lambda_n = \lambda_{-n}, \sum_{k=1}^{\infty} k|\Delta^2\lambda_k| + \sup_n |\lambda_n| < \infty\}$, where $\Delta^2\lambda_k = \Delta\lambda_k - \Delta\lambda_{k+1}$. It is well known that $q \subset bv$ and $q \subset (\widehat{L^1} \rightarrow \widehat{L^1})$ but $bv \not\subset (\widehat{L^1} \rightarrow \widehat{L^1})$. This result is significantly improved by finding an increasing family of sequence spaces dv_p between q and bv which are $(\widehat{L^1} \rightarrow \widehat{L^1})$ multipliers. Since the $(\widehat{L^1} \rightarrow \widehat{L^1})$ multipliers are the 2π -periodic measures, this result gives sufficient conditions for a sequence to be the Fourier coefficients of a measure.

Let L^1 be the space of all complex-valued 2π -periodic integrable functions and let $\widehat{L^1}$ be the space of all sequences $\hat{f}$ of Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

A sequence λ is an $(\widehat{L^1} \rightarrow \widehat{L^1})$ multiplier if $\lambda \cdot \hat{f} = (\lambda(n)\hat{f}(n))$ belongs to $\widehat{L^1}$ for every f in L^1 . Let C and M be the spaces of 2π -periodic continuous functions and 2π -periodic measures, respectively. It is well known that $(\widehat{L^1} \rightarrow \widehat{L^1}) = (\hat{C} \rightarrow \hat{C}) = (\hat{M} \rightarrow \hat{M}) = \hat{M}$ [3, vol. 2, p. 246].

The space of even sequences of bounded variation is defined by

$$bv = \left\{ \lambda \mid \lambda_n = \lambda_{-n}, \|\lambda\|_{bv} = \sum_{k=0}^{\infty} |\Delta\lambda_k| + \sup_n |\lambda_n| < \infty \right\},$$

where $\Delta\lambda_k = \lambda_k - \lambda_{k+1}$. The space of even bounded quasiconvex sequences is defined by

$$q = \left\{ \lambda \mid \lambda_n = \lambda_{-n}, \|\lambda\|_q = \sum_{k=1}^{\infty} k|\Delta^2\lambda_k| + \sup_n |\lambda_n| < \infty \right\},$$

where $\Delta^2\lambda_k = \Delta\lambda_k - \Delta\lambda_{k+1}$. We have $q \subset bv$ [1, vol. 2, p. 202].

It is also well known that $q \subset (\widehat{L^1} \rightarrow \widehat{L^1})$ [1, vol. 2, p. 203] but $bv \not\subset (\widehat{L^1} \rightarrow \widehat{L^1})$ [1, vol. 1, p. 239].

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It was shown in [2, §5] that the inclusion $q \subset (\widehat{L^1} \rightarrow \widehat{L^1})$ is essentially a consequence of the Cesàro summability of Fourier series in L^1 . Here we take advantage of the strong summability of Fourier series [6, vol. 2, p. 182] in order to significantly improve this multiplier result by finding an increasing family of sequence spaces between q and bv which are $(\widehat{L^1} \rightarrow \widehat{L^1})$ multipliers. These spaces are denoted by dv_p ($1 \leq p < \infty$). Since $(\widehat{L^1} \rightarrow \widehat{L^1}) = \hat{M}$, this result gives sufficient conditions for a series to be the Fourier series of a measure.

The space dv_1 is defined as follows. We write $\max_{2^n} = \max_{2^n \leq k < 2^{n+1}}$. The bv norm is $\|\lambda\|_{bv} = \sum_{k=0}^{\infty} |\Delta\lambda_k| + \sup_n |\lambda_n|$. If, in each dyadic block $2^n \leq k < 2^{n+1}$, we replace each $|\Delta\lambda_k|$ by its maximum $\max_{2^n} |\Delta\lambda_k|$ in that dyadic block, we obtain the dv_1 norm. Since there are 2^n terms in the n th dyadic block we have

$$\|\lambda\|_{dv_1} = \sum_{n=0}^{\infty} 2^n \max_{2^n} |\Delta\lambda_k| + \sup_n |\lambda_n|.$$

Then $dv_1 = \{\lambda \mid \lambda_n = \lambda_{-n}, \|\lambda\|_{dv_1} < \infty\}$ and clearly $dv_1 \subset bv$.

The spaces dv_p ($1 < p < \infty$) are obtained similarly using the $l^{p'}$ norm ($1/p + 1/p' = 1$) on the dyadic blocks instead of the maximum. Writing $\sum_{2^n} = \sum_{k=2^n}^{2^{n+1}-1}$, we have

$$\|\lambda\|_{dv_p} = \sum_{n=0}^{\infty} 2^{n/p} \left(\sum_{2^n} |\Delta\lambda_k|^{p'} \right)^{1/p'} + \sup_n |\lambda_n|$$

and $dv_p = \{\lambda \mid \lambda_n = \lambda_{-n}, \|\lambda\|_{dv_p} < \infty\}$.

It is natural to write $dv_{\infty} = bv$. The following theorem uses the Hölder inequality to show that the spaces dv_p increase with p .

THEOREM 1. $q \subset dv_1 \subset dv_p \subset dv_r \subset bv$ for $1 \leq p \leq r \leq \infty$.

PROOF. Every real sequence in q can be written as a difference of two positive bounded convex sequences [5]. Let λ be positive, bounded and convex. Then $\Delta\lambda_k$ is decreasing and $\sum \Delta\lambda_k < \infty$ [1, vol. 1, p. 5]. It follows that $\sum_n 2^n \max_{2^n} |\Delta\lambda_k| = \sum_n 2^n |\Delta\lambda_{2^n}| \leq 2 \sum_n |\Delta\lambda_n| < \infty$. The same holds for the real and imaginary parts of a complex sequence in q . This shows that $q \subset dv_1$. Let $\lambda \in dv_p$. Then by Hölder's inequality,

$$\begin{aligned} \sum_n 2^{n/r} \left(\sum_{2^n} |\Delta\lambda_k|^{r'} \right)^{1/r'} &\leq \sum_n 2^{n/r} (\sum_{2^n} 1)^{(1/p-1/r)} \left(\sum_{2^n} |\Delta\lambda_k|^{p'} \right)^{1/p'} \\ &= \sum_n 2^{n/p} \left(\sum_{2^n} |\Delta\lambda_k|^{p'} \right)^{1/p'} < \infty. \end{aligned}$$

This shows that $dv_p \subset dv_r$ for $1 < p \leq r < \infty$. The case $p = 1$ is similar, as well as the case $r = \infty$. $\square$

For each $f \in C$, denote the partial sums of the Fourier series by $s_n(x) = \sum_{k=-n}^n \hat{f}(k) e^{ikx}$ and define

$$\|f\|_{sp} = \sup_{x,n} \left\{ \frac{|s_0(x)|^p + \cdots + |s_n(x)|^p}{n+1} \right\}^{1/p}.$$

The sup norm on C is denoted by $\|f\|_{\infty} = \sup_x |f(x)|$.

THEOREM 2. For each $1 \leq p < \infty$, $\|\cdot\|_{sp}$ is a norm on C equivalent to $\|\cdot\|_\infty$.

PROOF. By [6, vol. 2, p. 182], for each $f \in C$ and each $p > 0$,

$$\frac{1}{n+1} \sum_{k=0}^n |s_k(x) - f(x)|^p \rightarrow 0$$

uniformly for all x as $n \rightarrow \infty$. Using Minkowski's inequality and

$$|f(x)| = \left\{ \frac{1}{n+1} \sum_{k=0}^n |f(x)|^p \right\}^{1/p}$$

we have $\|f\|_\infty \leq \|f\|_{sp} < \infty$ for all $f \in C$. Clearly $\|\cdot\|_{sp}$ is a norm on C . For each n ,

$$\sup_x \left\{ \frac{|s_0(x)|^p + \cdots + |s_n(x)|^p}{n+1} \right\}^{1/p}$$

is a continuous seminorm on C . By the uniform boundedness principle $\|\cdot\|_{sp} \leq M\|\cdot\|_\infty$ on C for some M . $\square$

Define

$$\|f\|_{dp} = \sup_{n,x} \left\{ \frac{|s_{2^n}(x)|^p + \cdots + |s_{2^{n+1}-1}(x)|^p}{2^n} \right\}^{1/p}.$$

We have $2^{-1/p}\|\cdot\|_{sp} \leq \|\cdot\|_{dp} \leq 3^{1/p}\|\cdot\|_{sp}$ and hence $\|\cdot\|_{dp}$ is also equivalent to $\|\cdot\|_\infty$ on C for each $1 \leq p < \infty$.

To show that $dv_p \subset (\widehat{L^1} \rightarrow \widehat{L^1})$ for all $1 \leq p < \infty$ we start with $p = 1$ and prove in Theorem 3 that $dv_1 \cdot \hat{C} = \hat{C}$, where $\cdot$ denotes the coordinatewise product. Then, using the same pattern, we prove the more general theorem $dv_p \cdot \hat{C} = \hat{C}$ for $1 < p < \infty$.

THEOREM 3. $dv_1 \cdot \hat{C} = \hat{C}$.

PROOF. Since $e = (\dots, 1, 1, 1, \dots)$ is an element of dv_1 , it is sufficient to show that $dv_1 \cdot \hat{C} \subset \hat{C}$. Let $f \in C$ and $\mu \in dv_1$. We have $\mu = \lambda + ce$ for some complex number c and a null sequence $\lambda \in dv_1$. It is sufficient to show $\lambda \cdot \hat{f} \in \hat{C}$. For technical reasons and without loss of generality we assume $\hat{f}(0) = 0$. Define the Cesàro partial sums

$$\sigma^r \lambda f(x) = \sum_{k=-r}^{+r} \left(1 - \frac{|k|}{r+1}\right) \lambda_k \hat{f}(k) e^{ikx}.$$

We show that the Cesàro partial sums $\sigma^r \lambda f(x)$, with $r = 2^m$ ($m = 1, 2, \dots$), form a Cauchy sequence in C . Using summation by parts we obtain

$$\begin{aligned} \sigma^r \lambda f(x) &= \sum_{k=0}^r \left(1 - \frac{k}{r+1}\right) \lambda_k (\hat{f}(k) e^{ikx} + \hat{f}(-k) e^{-ikx}) \\ &= \sum_{k=0}^r \left\{ \left(1 - \frac{k}{r+1}\right) \Delta \lambda_k + \frac{1}{r+1} \lambda_{k+1} \right\} s_k(x). \end{aligned}$$

Let $\varepsilon > 0$. For $r = 2^m < 2^n = s$, we have

$$\begin{aligned}
 & \sigma^s \lambda f(x) - \sigma^r \lambda f(x) \\
 &= \sum_{k=0}^r \left\{ \left(\frac{k}{r+1} - \frac{k}{s+1} \right) \Delta \lambda_k - \left(\frac{1}{r+1} - \frac{1}{s+1} \right) \lambda_{k+1} \right\} s_k(x) \\
 & \quad + \sum_{k=r+1}^s \left\{ \left(1 - \frac{k}{s+1} \right) \Delta \lambda_k + \frac{1}{s+1} \lambda_{k+1} \right\} s_k(x) \\
 &= \left(\frac{1}{r+1} - \frac{1}{s+1} \right) \sum_{k=0}^r (k \Delta \lambda_k - \lambda_{k+1}) s_k(x) \\
 & \quad + \sum_{k=r+1}^s \left(\left(1 - \frac{k}{s+1} \right) \Delta \lambda_k + \frac{1}{s+1} \lambda_{k+1} \right) s_k(x)
 \end{aligned}$$

and hence

$$\begin{aligned}
 & \| \sigma^s \lambda f(x) - \sigma^r \lambda f(x) \|_{\infty} \\
 & \leq \sup_x \left\{ \frac{1}{r+1} \sum_{k=0}^r (k |\Delta \lambda_k| + |\lambda_{k+1}|) |s_k(x)| \right. \\
 & \quad \left. + \sum_{k=r+1}^s \left(\left(1 - \frac{k}{s+1} \right) |\Delta \lambda_k| + \frac{1}{s+1} |\lambda_{k+1}| \right) |s_k(x)| \right\} \\
 & \leq \frac{1}{2^m} \sum_{j=0}^m \sup_x \sum_{2^j} (k |\Delta \lambda_k| + |\lambda_{k+1}|) |s_k(x)| \\
 & \quad + \sum_{j=m+1}^n \sup_x \sum_{2^j} \left(\left(1 - \frac{k}{s+1} \right) |\Delta \lambda_k| + \frac{1}{s+1} |\lambda_{k+1}| \right) |s_k(x)| \\
 & \leq \frac{1}{2^m} \sum_{j=0}^m \max_{2^j} (k |\Delta \lambda_k| + |\lambda_{k+1}|) 2^j \|f\|_{d_1} \\
 & \quad + \sum_{j=m+1}^n \max_{2^j} \left(|\Delta \lambda_k| + \frac{1}{s+1} |\lambda_{k+1}| \right) 2^j \|f\|_{d_1}.
 \end{aligned}$$

Choose M such that

$$\sum_{j=M+1}^{\infty} 2^j \max_{2^j} |\Delta \lambda_k| + \max_{k > 2^M} |\lambda_k| < \frac{\varepsilon}{6 \|f\|_{d_1}}$$

and let $m > M$ such that

$$\frac{2^{M+1}}{2^m} < \frac{\varepsilon}{3 \|f\|_{d_1} \|\lambda\|_{dv_1}}.$$

Then

$$\begin{aligned}
 & \|\sigma^s \lambda f(x) - \sigma^r \lambda f(x)\|_\infty \leq \\
 & \left(\frac{2^{M+1}}{2^m} \sum_{j=0}^M 2^j \max_{2^j} |\Delta \lambda_k| + \frac{1}{2^m} \sum_{j=0}^M 2^j \max_{2^j} |\lambda_k| \right) \|f\|_{d_1} \\
 & + \left(\sum_{j=M+1}^m 2^{j+1} \max_{2^j} |\Delta \lambda_{k+1}| + \sup_{k>2^M} |\lambda_{k+1}| \right) \|f\|_{d_1} \\
 & + \left(\sum_{j=m+1}^n 2^j \max_{2^j} |\Delta \lambda_k| + 2 \sup_{k>2^m} |\lambda_{k+1}| \right) \|f\|_{d_1} \\
 & < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.
 \end{aligned}$$

Let e^k be the sequence with 1 in the k th position and 0 elsewhere. Since $\hat{C}$ is complete with respect to the norm $\|\hat{f}\| = \|f\|_\infty$ and has continuous coordinate functionals $\hat{f} \rightarrow \hat{f}(k)$, $k = \pm 1, \pm 2, \dots$, it follows that $\sum_{k=-r}^{+r} (1 - |k|/(r+1)) \lambda_k \hat{f}(k) e^k$ converges to $\lambda \cdot \hat{f}$ in $\hat{C}$. Hence $\lambda \cdot \hat{f} \in \hat{C}$. $\square$

THEOREM 4. For $1 < p < \infty$, $dv_p \cdot \hat{C} = \hat{C}$.

PROOF. We follow the pattern of the proof of Theorem 3 and use Hölder's inequality. It is sufficient to show that $\lambda \cdot \hat{f} \in \hat{C}$ for every null sequence λ in dv_p and every f in C . Let $r = 2^m < 2^n = s$ and $1/p + 1/p' = 1$.

$$\begin{aligned}
 & \|\sigma^s \lambda f(x) - \sigma^r \lambda f(x)\|_\infty \\
 & \leq \frac{1}{2^m} \sum_{j=0}^m \sup_x \sum_{2^j} (k |\Delta \lambda_k| + |\lambda_{k+1}|) |s_k(x)| \\
 & + \sum_{j=m+1}^n \sup_x \sum_{2^j} \left(\left(1 - \frac{k}{s+1}\right) |\Delta \lambda_k| + \frac{1}{s+1} |\lambda_{k+1}| \right) |s_k(x)| \\
 & \leq \frac{1}{2^m} \sum_{j=0}^m \left\{ 2^{j+1} \left(\sum_{2^j} |\Delta \lambda_k|^{p'} \right)^{1/p'} + \max_{2^j} |\lambda_{k+1}| \right\} 2^{j/p} \|f\|_{d_p} \\
 & + \sum_{j=m+1}^n \left\{ \left(\sum_{2^j} |\Delta \lambda_k|^{p'} \right)^{1/p'} + \frac{2^{j/p'}}{s+1} \max_{2^j} |\lambda_{k+1}| \right\} 2^{j/p} \|f\|_{d_p} \\
 & \leq \|f\|_{d_p} \left\{ \frac{2^{M+1}}{2^m} \left(\sum_{j=0}^M 2^{j/p} \left(\sum_{2^j} |\Delta \lambda_k|^{p'} \right)^{1/p'} + \sup |\lambda_{k+1}| \right) \right. \\
 & + \left(2 \sum_{j=M+1}^m 2^{j/p} \left(\sum_{2^j} |\Delta \lambda_k|^{p'} \right)^{1/p'} + \sup_{k>2^M} |\lambda_{k+1}| \right) \\
 & \left. + \left(\sum_{j=m+1}^n 2^{j/p} \left(\sum_{2^j} |\Delta \lambda_k|^{p'} \right)^{1/p'} + 2 \sup_{k>2^m} |\lambda_{k+1}| \right) \right\}.
 \end{aligned}$$

As in Theorem 3, this can be made arbitrarily small by choosing a sufficiently large M and m .

Finally, since $(\hat{C} \rightarrow \hat{C}) = (\widehat{L^1} \rightarrow \widehat{L^1}) = \hat{M}$, we have the following result.

THEOREM 5. *For $1 \leq p < \infty$, we have $dv_p \subset (\hat{C} \rightarrow \hat{C}) = (\widehat{L^1} \rightarrow \widehat{L^1}) = \hat{M}$.*

N. Tanović-Miller has pointed out to me that Fomin showed in [4] that for $1 < p < \infty$, every null sequence in dv_p is in $\widehat{L^1}$. This can be derived from Theorem 5 and vice versa. Fomin's proof uses a theorem of Telyakovskii on the integrability of trigonometric series which in turn uses an integrability condition of Boas. The proof of Theorem 5 is quite direct using the strong summability of Fourier series.

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