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NONEXISTENCE OF EQUIVARIANT DEGREE ONE MAPS

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ABSTRACT. We establish the nonexistence of equivariant maps between some classes of manifolds carrying G-free actions and spheres, whose degree is prime to $|G|$, the order of G, where G is a finite nontrivial group.

Dold, in his article [**D**], proved the following theorem:

If a map $f\colon S^m \to S^n$ commutes with some free action of a finite nontrivial group G on the spheres S^m, S^n, then $n \geq m$.

We shall prove that this result remains true for all G-free actions on closed oriented manifolds and G-maps $f\colon M^m \to N^n$ under the following hypothesis:

(H) Assume that S^m has a free G-action and that there is a G-map $h\colon M^m \to S^m$ such that the degree of h is prime to $|G|$.

We use the notion of the trace of an action.

For all G-spaces X, Gottlieb [**G**] defined an integer $\operatorname{tr}(G, X)$, called the trace of the action, having several properties. For our purposes, we select the following:

G_1: If G is a finite group of order $|G|$ acting on a manifold N which is dominated by a finite CW complex, then $\operatorname{tr}(G, N) = |G|$ if and only if the action is free.

G_2: If M^m and N^m are closed oriented compact G-manifolds where G acts by orientation preserving homeomorphisms, and $f, g\colon M^m \to N^m$ are G-maps, then $\operatorname{tr}(G, M)$ divides the coincidence number $L(f, g)$.

In fact, if $\deg f$ or $\deg g$ is not zero, G_2 can be generalized as follows.

PROPOSITION 1. *If M^m and N^m are closed oriented compact G-free manifolds and $f, g\colon M \to N$ are equivariant maps such that $\deg f \neq 0$ or $\deg g \neq 0$, then $\operatorname{tr}(G, M)$ divides $L(f, g)$.*

PROOF. If G acts by orientation preserving homeomorphisms then the result is nothing but G_2, proved by Gottlieb [**G**].

Now suppose that $\deg f \neq 0$. So if $h \in G$ reverses orientation in M, then h also reverses orientation in N.

Let us define an action $G \times S^1 \to S^1$ by

$$\begin{cases} h(z) = \bar{z} & \text{if } h \text{ reverses orientation,} \\ h(z) = z & \text{if } h \text{ preserves orientation,} \end{cases}$$

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for $h \in G$ and $z \in S^1$, the unit circle in the complex plane, and consider the G-free manifolds $M \times S^1$ and $N \times S^1$, and the equivariant maps

$$f \times \mathrm{id}, \qquad g \times c \colon M \times S^1 \to N \times S^1$$

where the action of G on $M \times S^1$ and $N \times S^1$ is the diagonal action, id is the identity function, and c is the constant function, whose image is $1 \in S^1$.

Since $M \times 1 \subset M \times S^1$ is an equivariant retract it follows that $\mathrm{tr}(G, M \times S^1) = \mathrm{tr}(G, M)$ (see [**G**]); also since G acts freely by orientation preserving homeomorphisms in $M \times S^1$ and $N \times S^1$ it follows from G_2 that $\mathrm{tr}(G, M \times S^1)$ divides $L(f \times \mathrm{id}, g \times c)$.

But an easy computation shows that $L(f \times \mathrm{id}, g \times c) = L(f, g)$. So $\mathrm{tr}(G, M)$ divides $L(f, g)$. Q.E.D.

Since in this note we assume that all manifolds are closed, compact, oriented, connected G-manifolds, where G acts freely it follows that if $f, g \colon M^m \to N^m$ are G-maps, then $|G|$ divides $L(f, g)$. This result was also obtained by Nakaoka [**N**, Theorem 4.1].

From the definition of $L(f, g)$ (see [**V**]), it follows that if $f, g \colon M^m \to S^m$ are maps, then $L(f, g) = \deg f + (-1)^m \deg g$.

PROPOSITION 2. *Let M^m and N^n be G-free manifolds and let M^m satisfy the hypothesis* (H). *If $f \colon M^m \to N^n$ is a G-map then $n \geq m$.*

PROOF. Let us assume that $m > n$. Since we assume that S^m is a G-free manifold, then S^m is m-universal for the group G, because it is $(m-1)$-connected [**St**]. So there is a G-map $h_1 \colon N^n \to S^m$. By hypothesis (H) we have also a G-map $h \colon M^m \to S^m$ whose degree is prime to $|G|$. Then we have two G-maps $(h_1 \circ f)$, $h \colon M^m \to S^m$ and $|G|$ divides $L(h_1 \circ f, h)$. But $L(h_1 \circ f, h) = \deg(h_1 \circ f) + (-1)^m \deg h$ and since $h_1 \circ f$ factors through a manifold of lower dimension we have that $\deg(h_1 \circ f) = 0$; we conclude that $|G|$ divides $\deg h$.

This implies G is trivial. Q.E.D.

Suppose now that the nontrivial finite group G acts freely on the closed oriented manifolds M^n, N^n, S^{m+n}.

Let us consider the diagonal action of G on the product $M^m \times N^n$. Since the projection to the first factor $\pi \colon M^m \times N^n \to M^m$ is equivariant, there is no equivariant map $f \colon M^m \times N^n \to S^{m+n}$ whose degree is prime to $|G|$. In particular we have

COROLLARY 3. *If the nontrivial finite group G acts freely on the closed oriented manifolds M^m, N^n, S^{m+n}, then there is no degree one equivariant map $f \colon M^m \times N^n \to S^{m+n}$, where G acts diagonally on $M \times N$.*

As another application let us consider the Stiefel manifold $V_k(R^n)$, the set of k-orthonormal frames of R^n, and the involution $\alpha \colon V_k(R^n) \to V_k(R^n)$ defined by $(e_{i_1}, \ldots, e_{i_k}) = (-e_{i_1}, \ldots, -e_{i_k})$.

Consider also the involution on S^m given by the antipodal map.

COROLLARY 4. *For $k > 1$ there is no equivariant odd degree map $f \colon V_k(R^n) \to S^m$, where $m = \dim V_k(R^n)$.*

PROOF. For $j < k$ the projection $\pi \colon V_k(R^n) \to V_j(R^n)$ is an equivariant map. Since $\dim V_j(R^n) < \dim V_k(R^n)$ the hypothesis (H) can not be satisfied. Q.E.D.

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