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EMBEDDINGS OF DIFFERENTIAL OPERATOR RINGS AND GOLDIE DIMENSION

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ABSTRACT. The differential operator ring $S = R[x;\delta]$ can be embedded in $A_1(R)$, the first Weyl algebra over R, where R is a $\mathbf{Q}$-algebra and δ is a locally nilpotent derivation on R. Furthermore $A_1(R)$ is again a differential operator ring over the image of S. We consider similar embeddings of the smash product $R\#U(L)$, where L is a finite dimensional Lie algebra and $U(L)$ is its universal enveloping algebra. We show that the Weyl algebra over R has the same Goldie dimension as R itself and use this to prove that R and $R[x;\delta]$ have the same Goldie dimension where R is again a $\mathbf{Q}$-algebra and δ is locally nilpotent.

Introduction. Let δ be a derivation on the ring R. The corresponding differential operator ring $S = R[x;\delta]$ is an associative ring formed by taking polynomials in x over R under the usual addition and with multiplication subject to the rule $xr - rx = \delta(r)$ for all $r \in R$. In Theorem 2 we show that if R is a $\mathbf{Q}$-algebra and δ is locally nilpotent then $S = R[x;\delta]$ can be embedding in

$$A_1(R) = \frac{R\langle X, Y\rangle}{\langle XY - YX - 1\rangle} = R[Y][X; d/dY],$$

the first Weyl algebra over R. This result is well known. For example it follows from an old result of Nouaze and Gabriel [**4**, Théorème 1.2] which gives conditions for $S = R[x;\delta]$ itself to be isomorphic to a Weyl algebra. An explicit embedding, similar to that used here, has been found by T. Masuda [**3**, Lemma 3.4].

This embedding is then used to show that if L is a finite dimensional nilpotent Lie algebra over a field k, of characteristic zero, which acts on R as locally nilpotent derivations, then the smash product $R\#U(L)$ embeds in $A_n(R)$, where $U(L)$ is the universal enveloping algebra of L and $n = \dim_k L$. The existence of such an embedding follows from [**1**, Corollary 4.4]. We also consider finite dimensional solvable Lie algebras acting as derivations and give a similar embedding with $A_n(R)$ replaced by a certain extension ring, $\bar{A}_n(R)$.

It was shown by R. Shock [**5**] that R and $R[x]$ have the same right Goldie dimension. Here we show that $\dim R = \dim A_1(R)$ and we use this to show that $\dim R[x;\delta] = \dim R$, where R is a $\mathbf{Q}$-algebra and δ is locally nilpotent.

Differential operator rings. Let δ be a locally nilpotent derivation on the $\mathbf{Q}$-algebra R and let $S = R[x;\delta]$ be the corresponding differential operator ring. Define $\eta\colon R \to R[Y] \subseteq A_1(R)$ by $\eta(r) = \tilde{r} = \sum_i (\delta^i(r)/i!)Y^i$. Since δ is locally nilpotent, this is a finite sum. Let $\tilde{R} = \eta(R)$.

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LEMMA 1. *$\eta\colon R \to R[Y] \subseteq A_1(R)$ is a ring embedding with $\eta(1) = 1$. Furthermore Y is transcendental over $\tilde{R}$ and $\tilde{R}[Y] = R[Y]$.*

PROOF. If $r, s \in R$, then, $\eta(r+s) = \eta(r)+\eta(s)$ since $\delta^n(r+s) = \delta^n(r)+\delta^n(s)$ for all n. The coefficient of Y^k in $\eta(r)\eta(s)$ is $\sum_{i=0}^k \delta^i(r)\delta^{k-i}(S)/i!(k-i)! = \delta^k(rs)/k!$, by Leibniz's rule. Thus $\eta(r)\eta(s) = \eta(rs)$. Note that the constant term of $\tilde{r}$ is r so that η is injective and, since $\delta(1) = 0$, $\eta(1) = 1$.

Now suppose $\tilde{r}_0 + \tilde{r}_1 Y + \cdots + \tilde{r}_t Y^t = 0$ and consider this as a polynomial in Y with coefficients from R. The constant term is r_0, so that $r_0 = 0$. If we know that $r_0, \ldots, r_{i-1}$ are all zero, then the coefficient of Y^i is r_i, giving that r_i is zero also. Thus $r_0 = \cdots = r_t = 0$, so that Y is transcendental over $\tilde{R}$. Clearly, $\tilde{R}[Y] \subseteq R[Y]$. Conversely, $Y \in \tilde{R}[Y]$ and, if $r \in R$, then $r = \tilde{r} - \widetilde{\delta(r)}Y + \cdots + ((-1)^n/n!)\widetilde{\delta^n(r)}Y^n$, where $\delta^{n+1}(r) = 0$, and hence $r \in \tilde{R}[Y]$. Thus $R[Y] \subseteq \tilde{R}[Y]$ so that equality holds. □

If A is any associative ring and $\alpha \in A$, then $\mathrm{ad}_\alpha(\beta) = [\alpha, \beta] = \alpha\beta - \beta\alpha$ for all $\beta \in A$. Note that if $r \in R$, we have $\tilde{r}, X \in A_1(R)$ and

$$\mathrm{ad}_X(\tilde{r}) = [X, \tilde{r}] = \sum_i (\delta^i(r)/i!)[X, Y^i] = \sum_i (\delta^i(r)/i!)iY^{i-1} = \widetilde{\delta(r)}.$$

We can now prove

THEOREM 2. *Let δ be a locally nilpotent derivation on the $\mathbf{Q}$-algebra R. Then $S = R[x;\delta]$ is isomorphic to $\tilde{R}[X]$ via the map taking r to $\tilde{r}$ and x to X. Furthermore $Y \in A_1(R)$ is transcendental over $\tilde{R}[X]$, $\tilde{R}[X]$ is invariant under ad_Y, and $A_1(R) = \tilde{R}[X][Y]$ is a differential operator ring over $\tilde{R}[X]$ with the locally nilpotent derivation $\mathrm{ad}_Y = -d/dX$.*

PROOF. Since $\tilde{R} \subset R[Y]$, X is transcendental over $\tilde{R}$ and, from above, $[X, \tilde{r}] = \widetilde{\delta(r)}$. Thus $\tilde{R}[X]$ is isomorphic to $R[x;\delta]$.

Now suppose $\sum_{i=0}^k f_i Y^i = 0$ where $f_i \in \tilde{R}[X]$ and $f_j \neq 0$ for some j. Let $l = \max_i X\text{-deg } f_i$ and assume l is minimal over all such equations. Applying ad_Y we have that $0 = [Y, \sum_{i=0}^k f_i Y^i] = -\sum_{i=0}^k (\partial f_i/\partial X)Y^i$. By the minimality of l, $\partial f_i/\partial X = 0$ for each i, so that $f_i \in \tilde{R}$. But Y is transcendental over $\tilde{R}$ by Lemma 1 and hence Y is also transcendental over $\tilde{R}[X]$.

Finally, since $\mathrm{ad}_Y = -d/dX$, it is clear that ad_Y is a locally nilpotent derivation when restricted to $\tilde{R}[X]$. Also $R[Y] \subseteq \tilde{R}[X][Y]$ and $X \in \tilde{R}[X] \subseteq \tilde{R}[X][Y]$. Thus $\tilde{R}[X][Y] = A_1(R)$. □

Now let $\bar{A}_1(R) = R[[Y]][X; d/dY]$, where $R[[Y]]$ is the ring of formal power series in Y over R and d/dY denotes formal differentiation of power series with respect to Y. Note that $A_1(R) = R[Y][X; d/dY] \subseteq \bar{A}_1(R)$.

If we drop the assumption that δ is locally nilpotent, $S = R[x;\delta]$ may still be embedded in $\bar{A}_1(R)$ when R is a $\mathbf{Q}$-algebra. Here we let $\eta\colon R \to R[[Y]]$ be given by $\eta(r) = \tilde{r} = \sum_i (\delta^i(r)/i!)Y^i$ where this sum is now allowed to be infinite. As before, η is a ring monomorphism with $\eta(1_R) = 1_R$.

It makes sense to take power series in Y with coefficients from $\tilde{R} \subseteq R[[Y]]$, since in the sum $\sum_{i=0}^\infty \tilde{r}_i Y^i$, the coefficient of each power of Y receives only finitely many contributions. We write the set of these elements $\tilde{R}[[Y]]$.

LEMMA 3. *In* $\tilde{R}[[Y]]$, $\sum_i \tilde{r}_i Y^i = \sum_i \tilde{s}_i Y^i$ *if and only if* $r_i = s_i$ *for each* i. *Furthermore* $\tilde{R}[[Y]] = R[[Y]]$.

PROOF. Suppose $\sum_i \tilde{r}_i Y^i = \sum_i \tilde{s}_i Y^i$ and consider both sides as power series in Y with coefficients from R. Comparing constant terms gives that $r_0 = s_0$. If we have verified that $r_i = s_i$ for $i = 0, 1, \ldots, n-1$, then considering the coefficients of Y^n gives that $r_n = s_n$. Thus by induction, $r_i = s_i$ for all i.

It is clear that $\tilde{R}[[Y]] \subseteq R[[Y]]$ and since the constant term of $\tilde{r}$ is r, it is easy to see that any element of $R[[Y]]$ can be inductively constructed as a power series in Y over $\tilde{R}$, so that the reverse inclusion holds. □

The following result is the analogue of Theorem 2 for $\bar{A}_1(R)$.

THEOREM 4. *Let* δ *be a derivation on the* $\mathbf{Q}$*-algebra* R. *Then* $S = R[x;\delta]$ *is isomorphic to* $\tilde{R}[X] \subseteq \bar{A}_1(R)$ *via the map taking* r *to* $\tilde{r}$ *and* x *to* X. *Furthermore* $\tilde{R}[[Y]][X] = R[[Y]][X]$.

PROOF. The isomorphism $R[x;\delta] \simeq \tilde{R}[X]$ is proved as before and since $\tilde{R}[[Y]] = R[[Y]]$, it follows that $\tilde{R}[[Y]][X] = R[[Y]][X]$. □

We remark that $R[[Y]][X] \neq R[X][[Y]]$ since the elements of $R[[Y]][X]$ have finite X-degree. It can be shown that $R[[Y]][X] \subseteq R[X][[Y]]$ and consists of the elements of $R[X][[Y]]$ with finite X-degree.

If R is a $\mathbf{Q}$-algebra, it is known that the ideals of $A_1(R)$ are generated by their intersection with R [**2**, Satz 4.10]. Thus the map taking $I \triangleleft R$ to $A_1(I) \triangleleft A_1(R)$, is a bijection between the ideals of R and those of $A_1(R)$. If R is right Noetherian a similar result holds for $\bar{A}_1(R)$. First we require a lemma.

LEMMA 5. *Let* R *be a* $\mathbf{Q}$*-algebra and let* $I \subseteq R[[Y]]$ *be a finitely generated right ideal of* $R[[Y]]$. *Then* $I = (I \cap R)R[[Y]]$ *if and only if* I *is closed under* d/dY.

PROOF. If $I = (I \cap R)R[[Y]]$, it is clear that I is closed under d/dY.

Now suppose I is closed under d/dY and let $K_t = \{a_t \mid \sum_i a_i Y^i \in I\}$. Since I is closed under multiplication by Y, $K_i \subseteq K_{i+1}$. Also I is closed under d/dY and R is a $\mathbf{Q}$-algebra, so that $K_{i+1} \subseteq K_i$. Thus $K_t = K_0$ for all t and K_0 is a right ideal of R. It follows that $I \subseteq K_0[[Y]]$.

Let $\phi_1, \ldots, \phi_n$ generate I as a right ideal of $R[[Y]]$. Note that the constant terms of $\phi_1, \ldots, \phi_n$ must generate K_0 as a right ideal of R. A simple inductive argument now shows that $\phi_1, \ldots, \phi_n$ generate $K_0[[Y]]$ as a right ideal of $R[[Y]]$. Thus $I = K_0[[Y]]$ and since we have shown that K_0 is finitely generated as a right ideal of R,

$$I = K_0[[Y]] = K_0 R[[Y]] = (I \cap R)R[[Y]]. \quad \square$$

We now give the ideal correspondence between R and $\bar{A}_1(R)$ when R is Noetherian. If I is an ideal of R, let $\bar{A}_1(I)$ be the ideal of $\bar{A}_1(R)$ consisting of the elements of $\bar{A}_1(R)$ whose coefficients lie in I. If T is any ring, let $\mathcal{I}(T)$ denote the set of two-sided ideals of T.

LEMMA 6. *Let* R *be a right Noetherian* $\mathbf{Q}$*-algebra. Then* $\phi\colon \mathcal{I}(R) \to \mathcal{I}(\bar{A}_1(R))$, *where* $\phi(I) = \bar{A}_1(I)$, *is a bijection. This map preserves sums, products, and intersections. Also primes are sent to primes and primitives to primitives.*

PROOF. Suppose $J \triangleleft \bar{A}_1(R)$. We need to show $J = \bar{A}_1(I)$ for some $I \triangleleft R$. Let $\alpha = \sum_{i=0}^n f_i X^i \in J$, where $f_i \in R[[Y]]$. Note that $\mathrm{ad}_Y \alpha = -\sum_{i=1}^n f_i i X^{i-1} \in J$, and so, by induction on the X-degree of α, we can conclude that $f_i \in J$ for each $i > 0$, and then, that $f_0 \in J$. Thus $J = K[X]$, where $K = J \cap R[[Y]]$. K is finitely generated as a right ideal of $R[[Y]]$ since R, and hence $R[[Y]]$, is right Noetherian. Also K is invariant under $\mathrm{ad}_X = d/dY$ so that by Lemma 5, $K = I[[Y]]$ where $I = K \cap R$. Thus $J = K[X] = \bar{A}_1(I)$.

If $I, J \triangleleft R$, it is clear that $IJ = \phi(I)\phi(J) \cap R$. Thus $\phi(I)\phi(J) = \bar{A}_1(IJ) = \phi(IJ)$. Similarly $\phi(I+J) = \phi(I)+\phi(J)$ and $\phi(I \cap J) = \phi(I) \cap \phi(J)$. Now since ϕ preserves products and inclusions, it is clear that ϕ sends primes to primes. To show that primitives go to primitive, let $P \triangleleft R$ be the annihilator of the simple right R-module M and let $M[[Y]] = M \otimes_R R[[Y]]$. $R[[Y]]$ is a right $R[[Y]]$-module and this action can be extended to $\bar{A}_1(R)$ by letting $f \cdot X = -df/dY$ for each $f \in R[[Y]]$. It is easily checked that $R[[Y]]$ is now an $(R - \bar{A}_1(R))$-bimodule so that $M[[Y]] = M \otimes_R R[[Y]]$ becomes a right $\bar{A}_1(R)$-module. Fix m to be a nonzero element of M. Since M is simple any element of $M[[Y]]$ can be written in the form $m \otimes f$ for some $f \in R[[Y]]$. Let V be a nonzero $R[[Y]]$-submodule of $M[[Y]]$ and take I to be the right ideal of $R[[Y]]$ given by $I = \{f \in R[[Y]] | m \otimes f \in V\}$. $R[[Y]]$ is right Noetherian and V is an $\bar{A}_1(R)$-submodule of $M[[Y]]$, so that I is a finitely generated right ideal of $R[[Y]]$. Thus, by Lemma 5, I is generated as a right ideal of $R[[Y]]$ by its intersection with R. Let $J = I \cap R$. Then V is generated as an $\bar{A}_1(R)$-module by $\{m \otimes j | j \in J\} = \{mj \otimes 1 | j \in J\} = M \otimes 1$, since M is a simple R-module. Now $V = M[[Y]]$ so that $M[[Y]]$ is a simple $\bar{A}_1(R)$-module. Since the annihilator of $M[[Y]]$ is an ideal of $\bar{A}_1(R)$, it is of the form $\bar{A}_1(I)$ for some $I \triangleleft R$. Note that the right annihilator of $M = M \otimes 1 \subset M[[Y]]$ in R is P, so that $I \subset P$. Conversely, since P is finitely generated as a right ideal of R, it is clear that P annihilates $M[[Y]]$ and then, that $\bar{A}_1(P)$ is the annihilator of the simple module $M[[Y]]$. □

The following example, due to Passman, shows that Lemma 5 fails if I is not assumed to be finitely generated. This also shows that we cannot drop the Noetherian hypothesis in Lemma 6.

EXAMPLE 7. Let k be a field of characteristic zero and let

$$R = k + tk[x] \subseteq k[x, t | t^2 = 0].$$

Also let I be the ideal of $R[[Y]]$, invariant under d/dY, generated by te^{xY}. Then I is nonzero, $d/dY(I) \subseteq I$, but $R \cap I = 0$. Furthermore $I[X] \subseteq R[[Y]][X] = \bar{A}_1(R)$ is a nonzero ideal of $\bar{A}_1(R)$ but $I[X] \cap R = 0$.

PROOF. It is clear that R is a subring of $k[x, t | t^2 = 0]$ which does not contain x. Note that $d/dY(te^{xY}) = txe^{xY}$, so that $te^{xY}, txe^{xY}, tx^2e^{xY}, \ldots$ generate I as an ideal of $R[[Y]]$. Now suppose $r \in R \cap I$. Then $r = \sum_{i=0}^{n} tx^i e^{xY} f_i$, where $f_i \in R[[Y]]$. Consider this as an equation in $k[x, t | t^2 = 0][[Y]]$. Then $re^{-xY} = \sum_{i=0}^{n} x^i (tf_i)$. Since $t^2 = 0$, tf_i does not involve x, so that the right-hand side of this equation has bounded x-degree. If r is nonzero, the left-hand side has unbounded degree in x. Thus $r = 0$ and $I \cap R = 0$.

The last statement is clear. □

$\bar{A}_n(R)$ is defined inductively to be

$$\bar{A}_1(\bar{A}_{n-1}(R)) = R[[Y_1]][X_1; d/dY_1] \cdots [[Y_n]][X_n; d/dY_n].$$

Now let L be a Lie algebra over the field k and let R be a k-algebra. Then L is said to act as derivations on R if there is a Lie algebra map $\alpha \colon L \to \mathrm{Der}_k R$, where $\mathrm{Der}_k R$ is the Lie algebra of k-linear derivations on R. The smash product, $R\#U(L)$, of R with $U(L)$ the universal enveloping algebra of L, is the k-space $R \otimes_k U(L)$. It becomes a k-algebra under the multiplication coming from the subalgebras $R = R\otimes 1$ and $U(L) = 1\otimes U(L)$ together with the additional rule that if $x \in L$ and $r \in R$, then $[x,r] = xr - rx = \delta_x(r)$, where $\delta_x = \alpha(x)$.

THEOREM 8. *Let R be an algebra over a field k of characteristic zero and let L be a finite dimensional solvable Lie algebra over k which acts on R as derivations. Then the smash product $R\#U(L)$ can be embedded in $\bar{A}_n(R)$, where $n = \dim_k L$.*

PROOF. Since L is solvable, we can choose a basis $x_1, \ldots, x_n$ for L, where $\langle x_1, \ldots, x_i\rangle \lhd \langle x_1, \ldots, x_{i+1}\rangle$. Then $R\#U(L) = R[x_1;\delta_1][x_2;\delta_2]\cdots[x_n;\delta_n]$ is an iterated differential operator ring, where δ_{i+1} is a derivation on the ring $R[x_1;\delta_1]\cdots[x_i;\delta_i]$.

Now proceed by induction on n, the case $n = 1$ being done in Theorem 4. Letting $R_i = R[x_i;\delta_1]\cdots[x_i;\delta_i]$, note that by the inductive hypothesis, R_{n-1} embeds in $\bar{A}_{n-1}(R)$. If $R_{n-1} \subseteq \bar{A}_{n-1}(R)$, then

$$R_n = R_{n-1}[x_n;\delta_n] \subseteq \bar{A}_1(R_{n-1}) \subseteq \bar{A}_1(\bar{A}_{n-1}(R)) = \bar{A}_n(R). \quad \square$$

In the case where L is nilpotent and acts on R as locally nilpotent derivations, the successive derivations δ_i, which occur in the proof of the last theorem, are locally nilpotent. Thus we have the following result. The existence of such an embedding follows from [**1**, Corollary 4.4].

THEOREM 9. *Let R be an algebra over a field k of characteristic zero and let L be a finite dimensional nilpotent Lie algebra over k, which acts on R as locally nilpotent derivations. Then the smash product $R\#U(L)$ can be embedded in $A_n(R)$, where $n = \dim_k L$.* $\square$

EXAMPLE 10. Let $L = kx_1 + kx_2$ be the solvable Lie algebra of dimension 2, with $[x_2, x_1] = x_1$. Assume L acts on the k-algebra R, where the action of x_1 is given by δ_1 and the action of x_2 is δ_2. Then the embedding of $R^{\#}U(L)$ in $\bar{A}_2(R)$ is given by $r \in R$ goes to $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\delta_1^i\delta_2^j(r)/i!j!)Y_1^iY_2^j$, x_2 goes to X_2, and x_1 goes to $X_1e^{Y_2}$.

PROOF. Let $\bar{A}_1(R)$ have indeterminates X_1 and Y_1, while $\bar{A}_1(R[x_1;\delta_1])$ and $\bar{A}_1(\bar{A}_1(R))$ have indeterminates X_2 and Y_2. Also let $\varepsilon_1\colon R[x_1;\delta_1][x_2;\delta_2] \to \bar{A}_1(R[x_1;\delta_1])$ and $\varepsilon_2\colon R[x_1,\delta_1] \to \bar{A}_1(R)$ be the maps given in the proof of Theorem 8. We need to consider the composite

$$R[x_1;\delta_1][x_2;\delta_2] \xrightarrow{\varepsilon_2} \bar{A}_1(R[x_1;\delta_1]) \xrightarrow{\bar{A}_1(\varepsilon_1)} \bar{A}_1(\bar{A}_1(R)),$$

where $\bar{A}_1(\varepsilon_1)$ acts like ε_1 on $R[x_1;\delta_1]$ and sends X_2 and Y_2 to themselves. Thus if $r \in R$,

$$r \to \sum_{j=0}^{\infty}(\delta_2^j(r)/j!)Y_2^j \to \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(\delta_1^i\delta_2^j(r)/i!j!)Y_1^iY_2^j.$$

Also

$$x_1 \to \sum_{j=0}^{\infty}(\delta_2^j(x_1)/j!)Y_2^j = \sum_{j=0}^{\infty}(x_1/j!)Y_2^j = x_1e^{Y_2} \to X_1e^{Y_2}$$

and $x_2 \to X_2 \to X_2$. $\square$

We write $\dim R$ for the right Goldie dimension of the ring R. It was shown by R. Shock [**5**] that $\dim R[x] = \dim R$. We extend this in Theorem 15 to show that if R is a $\mathbf{Q}$-algebra and δ a locally nilpotent derivation on R, then $\dim R[x,\delta] = \dim R$. This was also shown to be true when R is a semiprime Goldie ring by G. Sigurdsson [**6**].

If I is a right ideal of R, we write $I[x;\delta]$ for $IR[x;\delta]$ which is a right ideal of $R[x;\delta]$. We use the symbol $\bar{\mathbf{r}}$ to denote right annihilators. Thus $\bar{\mathbf{r}}_R(t)$ is the right annihilator of t in R. It is convenient to isolate the argument in the following lemma.

LEMMA 11. *Let $S = R[x;\delta]$ be a differential operator ring and let $I \subseteq R[x;\delta]$ be a nonzero right ideal.*

(i) *If $h = c_0 + c_1x + \cdots + c_nx^n$, with $c_n \neq 0$, is a nonzero element of I of minimal degree n, then $\bar{\mathbf{r}}_S(h) = \bar{\mathbf{r}}_R(c_n)[x;\delta]$.*

(ii) *If $f = a_0 + \cdots + a_nx^n$ and $g = b_0 + \cdots + b_mx^m$ are nonzero elements of I with $a_n, b_m \neq 0$ and $n+m$ minimal subject to $fR[x;\delta] \cap gR[x;\delta] = 0$, then $\bar{\mathbf{r}}_S(f) = \bar{\mathbf{r}}_R(a_n)[x;\delta]$.*

PROOF. (i) Note that if $r \in R$, then $x^t r = rx^t + l$, where l involves only lower points of x. Thus $hr = c_nrx^n$+(lower degree terms). Now if $r \in \bar{\mathbf{r}}_R(c_n)$, then $hr \in I$ and hr has degree less than that of h. Hence $hr = 0$, and $\bar{\mathbf{r}}_R(c_n)[x;\delta] \subseteq \bar{\mathbf{r}}_S(h)$.

Conversely, let $k = b_0 + b_1x + \cdots + b_tx^t \in \bar{\mathbf{r}}_S(h)$. Note that $hk = c_nb_tx^{n+t}$+(lower degree terms). Since $hk = 0$, we find that $b_t \in \bar{\mathbf{r}}_R(c_n)$. Hence $hb_tx^t = 0$, giving that $b_0 + \cdots + b_{t-1}x^{t-1} \in \bar{\mathbf{r}}_S(h)$. Repeating this argument gives that $b_i \in \bar{\mathbf{r}}_R(c_n)$ for each i. Thus $k \in \bar{\mathbf{r}}_R(c_n)[x;\delta]$ and $\bar{\mathbf{r}}_S(h) \subseteq \bar{\mathbf{r}}_R(c_n)[x;\delta]$.

(ii) $fR[x;\delta]$ is a right ideal of $R[x;\delta]$ and since $n+m$ is minimal subject to $fR[x;\delta] \cap gR[x;\delta] = 0$, it follows that f is an element of $fR[x;\delta]$ of minimal degree. Now part (i) applies to give the result. □

The following lemma forms part of the proof of Shock's theorem and is included for the sake of completeness.

LEMMA 12. *Let $U \subseteq R$ be a uniform right ideal. Then $U[x] \subseteq R[x]$ is again a uniform right ideal.*

PROOF. If $u[x]$ is not a uniform right ideal, choose $f, g \in U[x]\backslash 0$, such that $\deg f + \deg g$ is minimal subject to $fR[x] \cap gR[x] = 0$. Let $f = a_0 + \cdots + a_nx^n$, with $a_n \neq 0$, and let $g = b_0 + \cdots + b_mx^m$, with $b_m \neq 0$. We can assume $m \leq n$ and, since U if uniform, we may further assume $a_n = b_m$. Then by Lemma 11(ii), $\bar{\mathbf{r}}_{R[x]}(f) = \bar{\mathbf{r}}_R(a)[x] = \bar{\mathbf{r}}_{R[x]}(g)$.

Now consider $h = f - gx^{n-m} \in U[x]$. If $h = 0$ then $f = gx^{n-m} \in fR[x] \cap gR[x]$ gives a contradiction. Thus $h \neq 0$. Note that $\deg h < \deg f$, so by the minimality of sum of the degrees, $hR[x] \cap gR[x] \neq 0$. Now choose $\alpha, \beta \in R[x]$ so that $h\alpha = g\beta \neq 0$. Thus $(f - gx^{n-m})\alpha = g\beta$ or $f\alpha = g(x^{n-m}\alpha + \beta)$. If $f\alpha = 0$ then $g\alpha = 0$ so that $h\alpha = 0$. Since $h\alpha \neq 0$ we conclude that $0 \neq f\alpha = g(x^{n-m}\alpha + \beta)$, which is a contradiction. □

The next result is well known.

LEMMA 13. $\dim R[x;\delta] \geq \dim R$ *for any differential operator ring* $R[x;\delta]$.

PROOF. If $I_1, \ldots, I_n$ is an independent set of right ideals of R, then $I_1[x;\delta], \ldots, I_n[x;\delta]$ forms an independent set of right ideals of $R[x;\delta]$. □

THEOREM 14. $\dim A_1(R) = \dim R$ *for any ring* R.

PROOF. From Lemma 13, we may assume $\dim R$ is finite. Let $\dim R = n$ and let $E = U_1 + \cdots + U_n$ be a direct sum of n uniform right ideals of R, which is essential as a right ideal. We claim that $EA_1(R) = U_1A_1(R) + \cdots + U_nA_1(R)$ is a direct sum of n uniform right ideals of $A_1(R)$ which is essential as a right ideal. Indeed $EA_1(R)$ is essential as an R-submodule of $A_1(R)$ and hence also as a right ideal. It remains to show that if $U \subseteq R$ is a uniform right ideal of R, then $UA_1(R)$ is a uniform right ideal of $A_1(R)$. This is achieved by a variation on the argument in Lemma 12.

Suppose $UA_1(R)$ is not uniform. Then let $f = \sum_{i=0}^{n} f_iX^i, g = \sum_{j=0}^{k} g_jX^j$, with $f_i, g_j \in U[Y]$, be nonzero elements of $UA_1(R)$ such that $fA_1(R) \cap gA_1(R) = 0$. Fix n and k so that $n+k$ is minimal and assume $n \geq k$. Note that if $h(Y) \in R[Y]$ then $X^nh(Y) = h(Y)X^n$+(lower powers of X). Also $U[Y]$ is a uniform ideal of $R[Y]$ by Lemma 12. Thus we may further assume $f_n = g_k$. Lastly we may also assume that the Y-degree of f_n is minimal in $f_nR[Y]$. Now let $a \in R$ be the leading coefficient of $f_n \in R[Y]$. Then $\bar{\mathbf{r}}_{A_1(R)}(f) = \bar{\mathbf{r}}_{R[Y]}(f_n)A_1(R)$ by Lemma 11(ii). But $\bar{\mathbf{r}}_{R[Y]}(f_n) = \bar{\mathbf{r}}_R(a)R[Y]$ by Lemma 11(i), so that $\bar{\mathbf{r}}_{A_1(R)}(f) = \bar{\mathbf{r}}_R(a)A_1(R)$. Similarly, $\bar{\mathbf{r}}_{A_1(R)}(g) = \bar{\mathbf{r}}_R(a)A_1(R)$. Since R commutes with X it follows that $\bar{\mathbf{r}}_R(a)A_1(R)$ annihilates gX^{n-k}.

Let $h = f - gX^{n-k}$. We may assume $h \neq 0$ since otherwise $f = gX^{n-k}$. Now since X-deg $h <$ X-deg f and $n+k$ is minimal, we see that $hA_1(R) \cap gA_1(R) \neq 0$. Choose $\alpha, \beta \in A_1(R)$ so that $0 \neq h\alpha = g\beta$. This implies that $f\alpha = g(X^{n-k}\alpha + \beta)$. If $f\alpha = 0$ then $h\alpha = 0$, since $\bar{\mathbf{r}}_{A_1(R)}(gX^{n-k}) \supseteq \bar{\mathbf{r}}_{A_1(R)}(g) = \bar{\mathbf{r}}_{A_1(R)}(f)$. Thus $0 \neq f\alpha = g(X^{n-k}\alpha + \beta)$ which contradicts the assumption that

$$fA_1(R) \cap gA_1(R) = 0. \qquad \square$$

Finally, we combine our methods to prove

THEOREM 15. *Let* R *be a* $\mathbf{Q}$*-algebra and let* δ *be a locally nilpotent derivation on* R. *Then* $\dim R[x;\delta] = \dim R$.

PROOF. We use the notation of Theorem 2 and combine that result with Lemma 13 to conclude that $\dim \tilde{R} \leq \dim \tilde{R}[X] \leq \dim \tilde{R}[X][Y]$. But $\tilde{R} \simeq R, \tilde{R}[X] \simeq R[x;\delta]$, and $\tilde{R}[X][Y] = A_1(R)$. Thus by Theorem 14, equality holds above. $\square$

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NOTE ADDED IN PROOF. Two papers are to appear on related subjects, *Uniform rank over differential operator rings and Poincaré-Birkhoff-Witt extensions*, Pacific J. Math., by A. Bell and and K. Goodearl, and *Goldie dimension of differential operator rings*, Comm. Algebra, by P. Grzeszczuk. Both these papers include additional situations in which $\dim R = \dim R[x;\delta]$. The first contains another proof of Theorem 14, with the result stated for any induced module, and gives an example which shows that Theorem 15 fails if δ is not assumed locally nilpotent.

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