ON THE GLOBAL DIMENSION OF A RING MODULO ITS NILPOTENT RADICAL

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ABSTRACT. For any preassigned integer $n \geq 3$, a Noetherian affine PI ring R is constructed with $\text{gl dim } R \leq 3$, but $\text{gl dim}(R/N(R)) = n$. A second similar ring is constructed with $\text{gl dim } R \leq 5$ and $\text{gl dim}(R/N(R)) = \infty$.

The question of whether $\text{rgl dim}(R/N(R)) \leq \text{rgl dim } R$ in a right Noetherian ring R , where $N(R)$ is the nilpotent radical of R , is an old one; see [ReSS, p. 294]. Eilenberg, Nagao, and Nakayama [ENN] provided examples to show that $\text{rgl dim}(R/I)$ can exceed $\text{rgl dim } R$ by an arbitrarily large amount for a nilpotent ideal I ($I \not\subseteq N(R)$) even in an Artinian hereditary ring; yet for Noetherian right hereditary rings, Small [S, Theorem 4] showed that $R/N(R)$ is hereditary, and results of Gordon and Small [GS] give the same result for more general hereditary rings (e.g., for a right hereditary ring with no infinite set of orthogonal idempotents). In these last two cases, the hereditary ring R is isomorphic to a triangular matrix ring. Fields [F2] showed that for any triangular matrix ring $R = \begin{bmatrix} R_1 & M \\ 0 & R_2 \end{bmatrix}$, $\text{rgl dim } R_i \leq \text{rgl dim } R$, so that a triangular matrix ring will not provide a counterexample to $\text{rgl dim}(R/N(R)) \leq \text{rgl dim } R$. In this paper we construct examples which show that for even a quite nice Noetherian ring R of finite global dimension, $\text{gl dim}(R/N(R))$ can be arbitrarily larger than $\text{gl dim } R$, or even infinite.

We begin by recalling the Eilenberg, Nagao, and Nakayama examples, which will be used in our construction.

EXAMPLE 1 [ENN]. Let $T = T_m(k)$ denote the ring of $m \times m$ upper triangular matrices over a field k , and $N(T)$ the ideal of strictly upper triangular matrices; then T is a hereditary ring and for $I = (N(T))^2$, $\text{gl dim}(T/I) = m - 1$. $\square$

EXAMPLE 2 [ENN]. Let

$$T = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a \end{bmatrix} \leq T_3(k)$$

and $I = (N(T))^2$. Then $\text{gl dim } T = 2$ but $\text{gl dim}(T/I) = \infty$. $\square$

We will produce rings R of finite global dimension by choosing them so that they are left subidealizers in a ring S of finite global dimension at an ideal A of R with $\text{gl dim}(R/A) < \infty$. In saying that R is a left subidealizer in S at A , we mean that R is a subring of S , A is a left ideal of S which is a (two-sided) ideal of R

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and $AS = S$. The following theorem guarantees that such subidealizers have finite global dimension.

THEOREM 3 ([RS OR McR, THEOREM 5.13]). *If R is a left subidealizer in S at A then $\text{rgl dim } R \leq \text{rgl dim } S + \text{rgl dim}(R/A) + 1$ and $\text{lgl dim } R \leq \text{lgl dim } S + \text{lgl dim}(R/A) + 1$. $\square$*

For a particular subidealizer we have previously calculated the global dimension exactly.

THEOREM 4 [KK]. *Let S be a ring and B be an ideal of S . Then*

$$\text{rgl dim} \begin{bmatrix} S & B \\ S & S \end{bmatrix} = \max\{\text{rgl dim } S, \text{rgl dim}(S/B) + 1\}. \quad \square$$

The rings we will construct are also fiber products (pullbacks). Let R_1 be a ring with ideal U_1 and R_2 be a ring with ideal U_2 such that $R_1/U_1 \simeq R_2/U_2$ under the isomorphism $\sigma: R_1/U_1 \rightarrow R_2/U_2$. Let R be the subring of $R_1 \oplus R_2$ defined by $R = \{(r_1, r_2): \sigma(r_1 + U_1) = r_2 + U_2\}$. We will call such a ring R the fiber product of R_1 and R_2 obtained by identifying R_1/U_1 with R_2/U_2 . Fiber products which can be shown to have finite global dimension because they are also subidealizers have previously been constructed by Robson [R] and Stafford [St].

Our examples will follow from the theorem below.

THEOREM 5. *Let T and S be rings with B an ideal of S and I an ideal of T such that $S/B \simeq T/I$. Consider the rings*

$$R_1 = \begin{bmatrix} T & 0 \\ T & T \end{bmatrix}, \quad R_2 = \begin{bmatrix} S & B \\ S & S \end{bmatrix},$$

and the ideals,

$$U_1 = \begin{bmatrix} T & 0 \\ T & I \end{bmatrix} \trianglelefteq R_1, \quad U_2 = \begin{bmatrix} S & B \\ S & B \end{bmatrix} \trianglelefteq R_2;$$

note that $R_1/U_1 \simeq R_2/U_2 \simeq S/B$. Let R be the fiber product of R_1 and R_2 obtained by identifying R_1/U_1 with R_2/U_2 . Then

(a) $\text{rgl dim } R \leq \max\{\text{rgl dim } T, \text{rgl dim } S\} + \text{rgl dim } T + 1$.

(b) If I is a nilpotent ideal of T , and S is a prime ring with $B \neq 0$ then $\text{rgl dim}(R/N(R)) = \max\{\text{rgl dim } R_2, \text{rgl dim}(T/N(T))\}$.

PROOF. Since R is a left subidealizer in $\Sigma = M_2(T) \oplus M_2(S)$ at the left ideal $A = \begin{bmatrix} T & 0 \\ T & 0 \end{bmatrix} \oplus \begin{bmatrix} S & B \\ S & B \end{bmatrix}$ of Σ , by Theorem 3 we have $\text{rgl dim } R \leq \text{rgl dim } \Sigma + \text{rgl dim}(R/A) + 1$, so (a) follows since $R/A \simeq T$. Furthermore, under the conditions of (b), $N(R) = \begin{bmatrix} N(T) & 0 \\ T & I \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $R/N(R) \simeq T/N(T) \oplus R_2$. $\square$

EXAMPLE 6. For any preassigned integer $n \geq 3$, there exists a Noetherian affine PI ring R , module-finite over its center, with $K \dim R = 1$ and $\text{gl dim } R \leq 3$, but $\text{gl dim}(R/N(R)) = n$.

PROOF. Let S be the subring of $M_n(R)$,

$$S = \begin{bmatrix} k[x] & \cdots & k[x] \\ (x) & k[x] & \\ \vdots & \ddots & \ddots & \vdots \\ (x) & \cdots & (x) & k[x] \end{bmatrix}$$

where $k[x]$ is a commutative polynomial ring over a field k , and let B be the ideal of S ,

$$B = \begin{bmatrix} (x) & (x) & k[x] & \cdots & k[x] \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & k[x] \\ & & & \ddots & (x) \\ (x) & \cdots & & & (x) \end{bmatrix};$$

note that $S/B \simeq T/I$ where $T = T_n(k)$ and $I = (N(T))^2$. Using the rings and ideals of Theorem 5, by Example 1 and Theorem 4, $\text{gl dim } R_2 = \max\{1, (n-1) + 1\} = n$, so applying Theorem 5, $\text{gl dim } R \leq \max\{1, 1\} + 1 + 1 = 3$ and $\text{gl dim}(R/N(R)) = \max\{n, 0\} = n$. $\square$

EXAMPLE 7. There exists a Noetherian affine PI ring R , module-finite over its center, with $K \dim R = 1$, $\text{gl dim } R \leq 5$, but $\text{gl dim}(R/N(R)) = \infty$.

PROOF. Choose T and I as in Example 2. Let

$$S = \begin{bmatrix} (k[x])^* & k[x] & k[x] \\ (x) & k[x] & k[x] \\ (x^2) & (x) & (k[x])^* \end{bmatrix},$$

where $*$ means entries agree mod x (this ring is based on an example of K. Fields, who used power series instead of polynomials [F1]); S is a $k[x]$ -order in $M_3(k(x))$ so by [AG] $\text{gl dim } S = \max \text{gl dim}(S_m)$, where the maximum is taken over all maximal ideals m of $k[x]$. Using an argument as in [F1, p. 130], it is not hard to check that $\text{gl dim } S = 2$. Let B be the ideal of S ,

$$B = \begin{bmatrix} (x) & (x) & (x) \\ (x) & (x) & (x) \\ (x^2) & (x) & (x) \end{bmatrix}$$

and note that $S/B \simeq T/I$. Using the rings and ideals of Theorem 5, by Example 2 and Theorem 4 $\text{gl dim } R_2 = \infty$; applying Theorem 5 $\text{gl dim } R \leq \max\{2, 2\} + 2 + 1 = 5$ and $\text{gl dim}(R/N(R)) = \infty$. $\square$

We close by noting that both Examples 6 and 7 can be easily modified so that $N(R)$ is a prime ideal. Let R' be the fiber product of T and R_2 be obtained by identifying T/I with R_2/U_2 . Then $R \simeq \begin{bmatrix} T & 0 \\ T & R' \end{bmatrix}$, so $\text{gl dim } R' \leq \text{gl dim } R$ by [F2]. The ring R' has the same properties as R .

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