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VANISHING THEOREMS FOR V-MANIFOLDS

DONU ARAPURA

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ABSTRACT. Generalizations to Kähler V-manifolds, are given for the Kodaira-Nakano-Sommese and Le Potier vanishing theorems. The proof uses Hodge theory for open varieties.

C. P. Ramanujam [**R**] gave an elegant proof of the Kodaira-Nakano vanishing theorem, deducing it from the weak Lefschetz theorem. It is well known that the Lefschetz theorem follows from the fact that the complement of a hyperplane section of a smooth projective variety is affine and hence has no cohomology above its complex dimension. This paper originated from the observation that it is possible to deduce the Kodaira-Nakano vanishing theorem directly from this fact, by using Deligne's log complex. By refining this technique, it is possible to prove a V-manifold version of Kodaira-Nakano-Sommese's vanishing theorem [**Sa**, Proposition 1.12]. This is the main result of this paper. At the suggestion of the referee, I have also included a Le Potier type vanishing theorem.

After the first draft of this paper was completed, I received the preprint of Esnault and Viehweg [**EV**]. Their paper contains a number of ideas in common with the present one, however their goals are slightly different.

Let X be a V-manifold, i.e., a complex space whose only singularities are quotient singularities by a finite group. The sheaf of V-manifold differentials $\tilde{\Omega}^p_X = (\Omega^p_X)^{\vee\vee}$ where $^\vee$ denotes dual. Whenever $U \subseteq X$ is an open set of the form V/G, V smooth and G a finite group, then $\tilde{\Omega}^p(U) = (\Omega^p(V))^G$ by [**St**, Lemma 1.8]. A line bundle L is called k-ample if some positive power of it is generated by global sections and the fibers of the associated map to projective space have dimension at most k. In particular L is 0-ample if and only if it is ample in the usual sense.

THEOREM 1. *If X is a compact n-dimensional Kähler V-manifold and L is a k-ample line bundle, then for all p, q with $p + q > n + k$,*

$$H^q(X, \tilde{\Omega}^p_X \otimes L) = 0.$$

The proof will use some basic facts from the Hodge theory of open V-manifolds. A divisor $D = \sum D_i$ on X has V-normal crossings if for each open $U \subseteq X$ of the form V/G the pullback of D to V has normal crossings. For such a D let $\tilde{\Omega}^p(\log D) = i_*\Omega^p(\log D)$ where $i\colon X - X_{\text{sing}} \to X$.

THEOREM 2 (STEENBRINK [**St**]). *If X is a Kähler V-manifold and $D = \sum D_i$ is a V-normal crossing divisor, then*

(i) $H^i(X, \mathbf{C}) = \bigoplus_{p+q=i} H^q(X, \tilde{\Omega}^p)$,

(ii) $H^i(X - D, \mathbf{C}) = \bigoplus_{p+q=i} H^q(X, \tilde{\Omega}^p(\log D))$.

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Note. (ii) is not explicitly stated in [**St**] but it follows from [**St**, 1.18, 1.19; **D**, Scholie 8.2].

LEMMA 1. *Let X be an n-dimensional Kähler V-manifold and D a reduced V-normal crossing divisor. Then*

(i) $H^q(X, \tilde{\Omega}^p(D)) \cong H^{n-q}(X, \tilde{\Omega}^{n-p}(-D))^\vee$,

(ii) $H^q(X, \tilde{\Omega}^p(\log D)) \cong H^{n-q}(X, \tilde{\Omega}^{n-p}(\log D)(-D))^\vee$.

PROOF. For each $x \in X$ there exists an identification $\hat{O}_x = \mathbf{C}[[x_1 \cdots x_n]]^G$, with G a finite group acting linearly on $R = \mathbf{C}[[x_1 \cdots x_n]]$. For any R-module M, we have a natural isomorphism $\mathrm{Hom}_{R^G}(\mathbf{C}, M^G) \cong \mathrm{Hom}_R(\mathbf{C}, M)^G$. Thus we get an isomorphism of the derived functors

$$\mathrm{Ext}^i_{R^G}(\mathbf{C}, -^G) \cong \mathbf{R}^1(\mathrm{Hom}_R(\mathbf{C}, -)^G) \cong \mathrm{Ext}^i_R(\mathbf{C}, -)^G;$$

the latter isomorphism follows from the exactness of $(-)^G$ on the category of $\mathbf{C}$-vector spaces. Thus for any R-module M, $\mathrm{Ext}^i_{R^G}(\mathbf{C}, M^G) = 0$ whenever $\mathrm{Ext}^i_R(\mathbf{C}, M) = 0$. Therefore $\mathrm{depth}\, M^G \geq \mathrm{depth}\, M$. So that the local cohomology groups $H^i_m(M^G) = 0$ for $i < \mathrm{depth}\, M$ [**H1**, Theorem 3.8]. The ring R^G is Cohen-Macaulay, therefore by local duality [**H1**, Theorem 6.7] $\mathrm{Ext}^i_{R^G}(M^G, \omega_{R^G}) = 0$ for $i > n - \mathrm{depth}\, M$.

X is Cohen-Macaulay so by global duality $H^i(X, F) \cong \mathrm{Ext}^{n-i}(G, \omega_X)^\vee$ for any coherent sheaf F and $\omega_X = \tilde{\Omega}^n =$ the dualizing sheaf. Suppose F denotes $\tilde{\Omega}^p(D)$ (resp. $\tilde{\Omega}^p(\log D)$), and let $M = \Omega^p_R$ (resp. $\Lambda^p \Omega^1_R(dx_1/x_1 \cdots dx_r/x_r)$ where $x_1 \cdots x_r = 0$ is the equation of D). Then $M^G = F_x \otimes \hat{O}_x$. The depth of M equals n, since it is free, therefore

$$\mathrm{Ext}^i_{O_x}(F_x, \omega_x) \oplus \hat{O}_x = \mathrm{Ext}^i_{R^G}(M^G, \omega_{R^G}) = 0 \quad \text{for } i > 0.$$

Therefore $\underline{\mathrm{Ext}}^i_{O_x}(F, \omega_X) = 0$ for $i > 0$, so that the spectral sequence for Ext groups collapses to give isomorphisms $\mathrm{Ext}^i_{O_x}(F, \omega_X) = H^i(\underline{\mathrm{Hom}}_{O_x}(F, \omega_X))$. Thus $H^i(X, F) \cong H^{n-i}(\underline{\mathrm{Hom}}_{O_x}(F, \omega_X))^\vee$.

Let $i\colon U \to X$ be the open set of smooth points of X. Then there is an isomorphism $\Omega^{n-p}_U(-D) \to \underline{\mathrm{Hom}}(\Omega^p_U(D), \omega_U)$ induced by the perfect pairing $\Omega^{n-p}_U(-D) \otimes \Omega^p_U(D) \to \Omega^n_U = \omega_U$. Therefore

$$\tilde{\Omega}^{n-p}(-D) \cong i_* \Omega^{n-p}_U(-D) \cong i_* \underline{\mathrm{Hom}}(\Omega^p_U(D), \omega)U) \cong \underline{\mathrm{Hom}}(\Omega^p_X(D), \omega_X);$$

the first and last isomorphisms follow from the reflexiveness of these sheaves. Combining these isomorphisms with the result at the end of the last paragraph proves (i).

To prove (ii), one checks easily that there is a perfect pairing $\Omega^{n-p}_U(\log D) \otimes \Omega^p_U(\log D) \to \Omega^n_U(\log D) = \omega_U(D)$. The rest of the argument is similar to the one above. □

For a topological space S, we define the topological cohomological dimension $\mathrm{tcd}(S) := \max\{i \mid H^i(S, \mathbf{C}) = 0\}$.

LEMMA 2. *Let D be a reduced irreducible divisor on a Kähler V-manifold X^n. Then $H^q(X, \tilde{\Omega}_X^p(D)) = H^q(D, \tilde{\Omega}_D^p(D))$ for all p, q with $p + q > \operatorname{tcd}(X - D)$.*

PROOF. We will prove the dual statement

$$(*) \qquad H^q(X, \tilde{\Omega}_X^p(-D)) = H^{q-1}(D, \tilde{\Omega}_D^{p-1}(-D)) \quad \text{if } p+q < 2n - \operatorname{tcd}(X - D).$$

By Lemma 1 and Theorem 2

$$H^q(X, \tilde{\Omega}_X^p(\log D)(-D)) \cong H^{n-q}(X, \tilde{\Omega}_X^{n-p}(\log D)) \subseteq H^{2n-p-q}(X, \mathbf{C}).$$

Hence, if $p + q < 2n - \operatorname{tcd}(X - D)$ then

$$H^q(X, \tilde{\Omega}_X^p(\log D)(-D)) = 0.$$

$(*)$ will follow from this equation together with the long exact cohomology sequence associated to the Poincaré residue sequence:

$$0 \to \tilde{\Omega}_X^p \to \tilde{\Omega}_X^p(\log D) \to \tilde{\Omega}_D^{p-1} \to 0. \quad \square$$

LEMMA 3. *Let $f\colon X^n \to Y^m$ be a proper map of a V-manifold to a Stein space such that the dimension of all the fibers of f are less than or equal to k. Then $\operatorname{tcd}(X) \leq n + k$.*

PROOF. Since Y is Stein the Leray spectral sequence degenerates to give

$$H^{pq} = H^q(X, \tilde{\Omega}_X^p) \cong H^0(X, R^q f_* \tilde{\Omega}_X^p).$$

By hypothesis $R^q f_* \tilde{\Omega}_X^p = 0$ if $q > k$, therefore $H^{pq} = 0$ if $q > k$. Also $H^{pq} = 0$ if $p > n$ so that $H^{pq} = 0$ if $p + q > n + k$. There is a resolution of the constant sheaf

$$0 \to \mathbf{C}_X \to \tilde{\Omega}_X^1 \to \tilde{\Omega}_X^2 \to \cdots \to \tilde{\Omega}_X^n \to 0$$

from which we get a spectral sequence

$$E_1^{pq} = H^{pq} \Rightarrow H^{p+q}(X, \mathbf{C}).$$

If $p + q > n + k$ then $E_1^{pq} = 0$ therefore $H^{p+q}(X, \mathbf{C}) = 0$. $\square$

The next lemma is well known in the smooth case (see [**R** or **V**]).

LEMMA 4. *Let X^n be a V-manifold and L a line bundle such that a positive power L^m possesses a reduced irreducible V-normal crossing divisor D. Then there exists an m-sheeted finite cover $\pi\colon Y \to X$ ramified along D with the following properties:*

(i) *Y is a V-manifold and $\pi^* L$ possesses a reduced irreducible V-normal crossing divisor, namely $(\pi^* D)_{\mathrm{red}}$.*

(ii) *$\mathbf{Z}/m\mathbf{Z}$ acts on Y and $\tilde{\Omega}_X^p = (\pi_* \tilde{\Omega}_Y^p)^{\mathbf{Z}/m\mathbf{Z}}$.*

PROOF. Let $\{U_i\}$ be an open cover of X such that $U_i = V_i/G_i$ with each V_i isomorphic to an open set in $\mathbf{C}^n$ and $G_i \subset \mathrm{Gl}_n(\mathbf{C})$ a finite group. Assume that L is locally trivial with respect to $\{U_i\}$ and let $\{g_{ij}\}$ be its transition functions. Let $\{f_i\}$ be the local equations of D so that $f_i = f_j (g_{ij})^m$. Then the coverings

$$\pi_i\colon Y_i = \{(y_i, u_i) \in \mathbf{C} \times U_i \mid y_i^m = f_i(u_i)\} \to U_i$$

patch together

$$\pi_j^{-1}(U_i \cap U_j) \to \pi_i^{-1}(U_i \cap U_j) \quad \text{by } (y_j, u_j) \to (y_j g_{ij}(u_j), u_j)$$

to give a global covering $\pi\colon Y \to X$. The divisors $\{y_i = 0\}$ patch together to form a divisor $(= \tau^* D_{\mathrm{red}})$ of $|\pi^* L|$ on Y.

By refining the covering, we can assume that for a fixed i there are coordinates $x_1 \cdots x_n$ on V_i such that $x_1 =$ the pullback of f_i from U_i to V_i, and V_i is equal to the polydisc $\{(x_1 \cdots x_n) | \, \forall j, |x_j| < \varepsilon_j\}$. Let $W = \{(t, x_2 \cdots x_n) | \, t < \varepsilon_1^{1/m}$ and $|x_j| < \varepsilon_j$ for $j = 2, \ldots, n\}$ and let $q\colon W \to V_i$ be given by $q(t, x_2 \cdots x_n) = (t^m, x_2 \cdots x_n)$. Let $p_i\colon V_i \to V_i/G_i \cong U_i$ denote the quotient map. Then x_1 is invariant since it equals $p_i^* f_i$, therefore the action of G_i factors through an action on $\{(x_2 \cdots x_n) | \, |x_j| < \varepsilon\}$. Let $\varsigma = \exp(2\pi i/m)$. Then we define an action of $\mathbf{Z}/m\mathbf{Z} \times G_i$ on W by $(r, g)(t, x_2 \cdots x_n) = (\varsigma^r t, g(x_2 \cdots x_n))$ and the quotient can be identified with Y_i via the maps $s\colon W \to Y_i$, $s(t, x_2 \cdots x_n) = (t, p_i(t^m, x_2 \cdots x_n))$. Thus Y_i is a V-manifold and $\pi^* D_{\mathrm{red}}|_{Y_i}$ is a reduced irreducible V-normal crossing divisor since $s^*(\pi^* D_{\mathrm{red}}\,|_{Y_i}) = \{t = 0\}$ is smooth.

We can define an action of $\mathbf{Z}/m\mathbf{Z}$ on Y_i by $r(y_i, u_i) \to (\varsigma^r y_i, u_i)$, these actions patch to give an action of $\mathbf{Z}/m\mathbf{Z}$ on Y. The last assertion of (ii) follows from the equalities

$$(\tilde{\Omega}^p_{Y_i})^{\mathbf{Z}/m\mathbf{Z}} = (\Omega^p_W)^{\mathbf{Z}/m\mathbf{Z} \times G_i} = (\Omega^p_{V_i})^{G_i} = \tilde{\Omega}^p_{U_i}. \quad \square$$

PROOF OF THEOREM 1. Let L be a k-ample bundle on an n-dimensional compact Kähler V-manifold X. The proof will proceed by induction on $n - k$, the case $n - k = 0$ is automatically true, so we will assume that $n > k$. For an $m > 0$, L^m is generated by global sections and the dimension of the fibers of the induced map $f\colon X \to \mathbf{P}^N$ are all less than or equal to k. Let $U_1 \cdots U_r$ be an open covering by sets of the form V_i/G_i with V_i smooth and G_i finite. By applying Sard's theorem to the map $\bigcup\{(x, H) \in V_i \times (\mathbf{P}^N)^\vee \mid f(x) \in H\} \to (\mathbf{P}^N)^\vee$ we see that there exists a hyperplane H with $f^*(H)$ a reduced irreducible V-normal crossing divisor. Let $\pi\colon Y \to X$ be the m-sheeted covering branched along $f^*(H)$, constructed in Lemma 4. Then $D = \pi^* f^*(H)_{\mathrm{red}} \mid \pi^* L|$ is a reduced irreducible V-normal crossing divisor. Applying Lemma 3 to the map $f \circ \pi\colon Y - D \to \mathbf{P}^N - H$, we find $\operatorname{tcd}(Y - D) \le n + k$. Hence, by Lemma 2, if $p + q > n + k$ then $H^q(Y, \tilde{\Omega}^p_Y(D)) = H^q(D, \tilde{\Omega}^p_D(D))$. But by induction we can assume that the latter group vanishes. By Lemma 4 we have $\tilde{\Omega}^p_X = (\pi_* \tilde{\Omega}^p_Y)^{\mathbf{Z}/m\mathbf{Z}} \subseteq \pi_* \tilde{\Omega}^p_Y$; furthermore, the inclusion is split by the trace $\eta \to \frac{1}{m} \sum g \cdot \eta$. Therefore $H^q(X, \tilde{\Omega}^p_X \otimes L)$ is a direct summand of $H^q(X, \pi_* \tilde{\Omega}^p_Y(D)) = H^q(Y, \tilde{\Omega}^p_Y(D))$. Therefore $H^q(X, \tilde{\Omega}^p_X \otimes L) = 0$ when $p + q > n + k$. $\square$

The techniques of the proof can sometimes yield sharper results.

THEOREM 3. *Let $|D|$ be a very ample linear system on a projective V-manifold X. Suppose there exists irreducible divisors $D_1 \cdots D_r$ $(r \le n)$ in $|D|$ such that $\sum D_i$ forms a reduced V-normal crossing divisor and such that* $\operatorname{tcd}(X - D_1) < n$ *and* $\operatorname{tcd}(D_1 \cap D_2 \cap \cdots \cap D_{i-1} - D_i) < n - i + 1$ *for $i = 1, \ldots, r$. Then $H^q(X, \tilde{\Omega}^p_X(D)) = 0$ whenever $p + q > n - r$.*

PROOF. By Lemma 2 and Theorem 1, if $p + q > n - r$ then

$$H^q(X, \tilde{\Omega}^p_X(D)) = H^q(X, \tilde{\Omega}^p_{D_1}(D)) = \cdots = H^q(X, \tilde{\Omega}^p_{D_r}(D)) = 0. \quad \square$$

As a special case we can take $X = \mathbf{P}^n$ and $D_1 \cdots D_n$ to be hyperplanes in general position. Then $X - D_1$ and $D_1 \cap D_2 \cap \cdots \cap D_{i-1} - D_i$ are contractible; thus we get the well-known results $H^q(\mathbf{P}^n, \tilde{\Omega}^p_{\mathbf{P}^n}(1)) = 0$ for all $p + q > 0$.

A vector bundle E is called k-ample iff the tautological bundle $O_{\mathbf{P}(E)}(1)$ is a k-ample line bundle, in particular 0-ampleness corresponds to ampleness in the sense of Hartshorne [**H2**]. Further discussion of this concept can be found in [**So**]. Here we will prove a vanishing theorem for such bundles, generalizing similar results proved in [**L** and **So**].

THEOREM 4. *If E is a k-ample vector bundle on a compact n-dimensional Kähler V-manifold then for all p, q with $p+q > n+k+r\cdot(rkE-r)$*

$$H^q(X, \tilde{\Omega}^p_X \otimes \Lambda^r E) = 0.$$

PROOF. Let $\pi\colon G = \mathrm{Gr}_r(E) \to X$ be the Grassmann bundle of codimension r planes in E. Let Q denote the universal quotient bundle on G. Then the line bundle $L = \Lambda^r Q$ induces the Plücker embedding $f\colon G \to \mathbf{P}(E)$. Therefore $L = i^* O_{\mathbf{P}(E)}(1)$ is k-ample, so by Theorem 1, $H^q(G, \tilde{\Omega}^p_G \otimes L) = 0$ for $p+q > \dim G + k = n + r\cdot(rkE-r)+k$.

As in the proof of Corollary 3 of [**L**] we see that

$$R^q\pi_*(\Omega^p_{G/X} \otimes L) = \begin{cases} \Lambda^r E & \text{if } p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the Leray spectral sequence yields

$$H^q(G, \tilde{\Omega}^i_X \otimes \Omega^{p-i}_{G/X} \otimes L) = \begin{cases} H^q(X, \tilde{\Omega}^p_X \otimes \Lambda^r E) & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By working on a local trivialization of E of the form $\{V_i/G_i\}$, one can easily check that there is an exact sequence

$$0 \to \pi^*\tilde{\Omega}^1_X \to \tilde{\Omega}^1_G \to \Omega^1_{G/X} \to 0$$

and more generally (see 115.16 of [**H3**]) there is a filtration $\tilde{\Omega}^p_G = F^0 \supset F^1 \supset \cdots$ with $F^i/F^{i+1} = \tilde{\Omega}^i_X \otimes \Omega^{p-i}_{G/X}$. Hence there is a spectral sequence

$$E_1^{ij} = H^{i+j}(G, \tilde{\Omega}^i_X \otimes \Omega^{p-i}_{G/X} \otimes L) \Rightarrow H^{i+j}(G, \tilde{\Omega}^p_G \otimes L).$$

This degenerates to give an isomorphism

$$H^q(X, \tilde{\Omega}^p_X \otimes \Lambda^r E) = H^q(G, \tilde{\Omega}^p_G \otimes L). \quad \square$$

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907