A GENERALIZED CAPACITY AND A UNIQUENESS THEOREM ON THE DYADIC GROUP

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ABSTRACT. In this paper we shall introduce a generalized capacity and give a necessary and sufficient condition for a subset of the dyadic group to be a U -set for a certain class of Walsh series.

Wade [2] proved that a subset of the dyadic group is a U -set for τ_α^+ if and only if it is of α -capacity zero. We have already partially generalized Wade's theorem in [4].

A dyadic interval of rank n , I_n^p , is the set of all 0-1 series, $x = (t_1, t_2, \dots)$, such that $\sum_{k=1}^n t_k/2^k = p/2^n$. $I_n(x)$ is the dyadic interval of rank n which contains x . Hence I_0^0 is the dyadic group.

For convenience we shall write $x = (\sum_{k=1}^\infty t_k/2^k)^-$ if $\lim_{k \rightarrow \infty} t_k = 1$ and $x = (\sum_{k=1}^\infty t_k/2^k)$ otherwise. We refer the reader to [3 and 4] for details of the dyadic group.

A dyadic measure m is a real valued set function satisfying the additive law $m(I_n^p) = m(I_{n+1}^{2p}) + m(I_{n+1}^{2p+1})$ for $n = 0, 1, \dots$, $p = 0, 1, \dots, 2^n - 1$. When $\mu \equiv \sum_{k=0}^\infty \hat{\mu}(k)w_k(x)$ is an arbitrary Walsh series, set

$$m_\mu(I_n^p) = \lim_{N \rightarrow \infty} \int_{I_n^p} \sum_{k=0}^N \hat{\mu}(k)w_k(x) dx = 1/2^n \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(p/2^n)$$

for each dyadic interval I_n^p . It is easy to see that m_μ is a dyadic measure and

$$\hat{\mu}(k) = \int_{I_0^0} w_k(x)m_\mu(dx) \equiv \int_0^{1^-} w_k(x)m_\mu(dx) \quad \text{for all } k.$$

Conversely an arbitrary dyadic measure always has its Walsh Fourier expansion [3].

Let $\varepsilon = \{\varepsilon_n\}_n$ be a sequence of positive numbers such that

$$(1) \quad 1 > \varepsilon_0 > \varepsilon_1 > \dots > \varepsilon_n > \dots > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \varepsilon_n = 0.$$

Since $\sum_{k=2^n}^{2^{n+1}-1} w_k(x) = 2^n w_{2^n}(x)$ for $x \in I_n^0$ ($n = 0, 1, \dots$) and $= 0$ otherwise,

$$\alpha(x) = 1 + \sum_{n=0}^\infty \varepsilon_n \sum_{k=2^n}^{2^{n+1}-1} w_k(x)$$

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converges uniformly on each $I_0^0 \setminus I_n^0$ and $\alpha(x)$ is a positive and integrable function. It is obvious that

$$\int_0^{1^-} \alpha(x) dx = 1, \quad \alpha(x) = \text{constant on } I_n^1, \quad n = 1, 2, \dots, \quad \alpha(x) \geq \alpha(y),$$

if $x < y$, $\alpha(x) = \sum_{n=0}^{\infty} a_n \chi_{I_n^0}(x)$, where χ_I is the characteristic function of I , and $\sum_{n=0}^{\infty} a_n / 2^n = 1$.

For each Borel set $E \subseteq I_0^0$, $\mathfrak{M}(E)$ denotes the set of all Walsh Fourier series of nonnegative dyadic measures concentrated with E with total variation 1. For each $\mu \in \mathfrak{M}(E)$, the *potential function of μ* , U_μ , and the *energy of μ* , I_μ , are defined as follows:

$$U_\mu(x) = \int_0^{1^-} \alpha(x+t) m_\mu(dt) = \sum_{n=0}^{\infty} a_n m_\mu(I_n(x)),$$

$$\begin{aligned} I_\mu &= \int_0^{1^-} U_\mu(x) m_\mu(dx) = \sum_{n=0}^{\infty} a_n \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^2 \\ &= 1 + \sum_{n=0}^{\infty} \varepsilon_n \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2. \end{aligned}$$

$C(E) = 1/\inf_{\mu \in \mathfrak{M}(E)} \|U_\mu\|_\infty$ is called the ε -*capacity of E* . For a closed set E and $x \notin E$, there exists a dyadic interval $I_N(x)$ such that $I_N(x) \cap E = \emptyset$ and $m_\mu(I_N^p) = 0$ for $I_n^p \subseteq I_N(x)$. Hence we have

$$U_\mu(x) = \sum_{n=0}^{N-1} a_n m_\mu(I_n(x)) < \infty.$$

If $U_\mu(x) = \infty$, then $x \in E$. If there exists x_0 such that $U_\mu(x_0) = \infty$, then $\|U_\mu\|_\infty = \infty$.

We shall state some fundamental lemmas without proofs. The proofs of these lemmas are quite similar to those of theorems stated in the book of Kahane and Salem [1, pp. 31–37].

LEMMA 1. *For a closed subset of the dyadic group E we have*

- (i) $C(E) > 0$ if and only if there exists a Walsh series $\mu \in \mathfrak{M}(E)$ such that $I_\mu < \infty$ and
- (ii) $1/C(E) = \inf_{\mu \in \mathfrak{M}(E)} I_\mu \equiv I(E)$.

LEMMA 2. *If $I(E) < \infty$, there exists uniquely a dyadic measure m_{μ^*} , which is called the equilibrium dyadic measure of E , such that $\mu^* \in \mathfrak{M}(E)$ and $I_{\mu^*} = I(E)$.*

LEMMA 3. *If a nonnegative dyadic measure m_ν satisfies $I_\nu < \infty$, then, for a closed subset E , we have*

$$\begin{aligned} m_\nu(x \in E : U_{\mu^*}(x) < I(E)) \\ \equiv \lim_{n \rightarrow \infty} m_\nu \left(\bigcup I_n(x) : x \in E, U_{\mu^*}(x) < I(E) \right) = 0. \end{aligned}$$

Let $\mathcal{U}_\varepsilon^+$ be the set of all Walsh series of nonnegative dyadic measure μ satisfying

$$\sum_{n=0}^{\infty} \varepsilon_n \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}^*(k)|^2 < \infty$$

where $\varepsilon = \{\varepsilon_n\}_n$ satisfies (1). When $\varepsilon_n = 1/2^{n(1-\alpha)}$, $n = 0, 1, \dots$, $0 < \alpha < 1$, $\mathcal{U}_\varepsilon^+$ coincides with $\mathcal{T}_\alpha^+$ which was introduced by Wade in [2].

A subset of the dyadic group is said to be a U -set for $\mathcal{U}_\varepsilon^+$, if $\mu \in \mathcal{U}_\varepsilon^+$ and

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{except on } E$$

imply that μ is the zero series.

THEOREM 4. *A closed set of the dyadic group E is a U -set for $\mathcal{U}_\varepsilon^+$ if and only if the ε -capacity of E is zero.*

To prove Theorem 4 we need some lemmas.

LEMMA 5. *If E is of ε -capacity zero, then*

$$\lim_{N \rightarrow \infty} \inf_{\mu \in \mathfrak{M}(E_N)} \sup_{x \in E_N} |U_\mu(x)| = \infty$$

where $E_N = \bigcup_{x \in E} I_N(x)$.

PROOF. The conclusion follows from the definition of the ε -capacity.

LEMMA 6. *If E is of ε -capacity positive, then there exists $\mu \in \mathfrak{M}(E)$ such that $U_\mu \in L^\infty$ and $U_\mu(x) \geq 1/C(E)$ a.e. on E .*

PROOF. The Walsh series of the equilibrium dyadic measure of E satisfies the conclusion.

LEMMA 7. *If $\mu \in \mathcal{U}_\varepsilon^+$ and I is a dyadic interval, then there exist a dyadic measure m_γ such that $\gamma \in \mathcal{U}_\varepsilon^+$ and a positive number n_0 such that $\sum_{k=0}^{2^n-1} \hat{\gamma}(k) w_k(x) = \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x)$ on I and $= 0$ otherwise for $n > n_0$.*

PROOF. The following m_γ satisfies the conclusion: $m_\gamma(I_n^p) = m_\mu(I_n^p)$ if $I_n^p \subset I$ and $= 0$ otherwise.

The following lemma is Lemma 4 of [4].

LEMMA 8. *If $\mu \in \mathcal{U}_\varepsilon^+$ satisfies*

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) &= 0 && \text{a.e. outside } F, \\ \limsup_{n \rightarrow \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| &< \infty && \text{everywhere on } F^c \text{ except perhaps} \\ &&& \text{on a countable set} \end{aligned}$$

when F is a closed subset of the dyadic group, then for each dyadic interval $I \subset F^c$ there exists an integer N such that $\sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0$ on I for $n \geq N$.

PROOF OF THEOREM 4. Assume that E is not of ε -capacity zero. Then the equilibrium dyadic measure m_{μ^*} satisfies $\|U_{\mu^*}\|_\infty < \infty$. It is obvious that μ^* is

not the zero series and satisfies

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \hat{\mu}^*(k) w_k(x) &= 0 \quad \text{except on } E, \\ \sum_{k=0}^{2^n-1} \hat{\mu}^*(k) w_k(x) &= 2^n m_{\mu^*}(I_n(x)) \geq 0. \end{aligned}$$

Since $I_{\mu^*} \leq \|U_{\mu^*}\|_{\infty}$ it is also clear that I_{μ^*} is finite. Hence E is an M -set for $\mathcal{U}_{\varepsilon}^+$; this contradicts the hypothesis. Then E must be of ε -capacity zero.

To prove sufficiency let E be a closed set of ε -capacity zero and let $\mu \in \mathcal{U}_{\varepsilon}^+$. By the argument of [2, p. 314] it suffices to show $\hat{\mu}(0) = 0$. Set $E_n = \bigcup_{x \in E} I_n(x)$ for all N . Since E is closed, we have $E = \bigcap_N E_N$. For sufficiently large n ,

$$(3) \quad \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{except on } E_N.$$

From the hypothesis m_{μ} is a nonnegative dyadic measure; then $\hat{\mu}(0) \geq 0$. From (3) we have

$$\hat{\mu}(0) = |\hat{\mu}(0)| = \int_{E_N} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) dx \quad \text{for } n > N.$$

By Lemma 5 there exists μ_N^* such that

$$U_{\mu_N^*}(x) \geq \|U_{\mu_N^*}\|_{\infty} = \inf_{\mu \in \mathfrak{M}(E_N)} \|U_{\mu}\|_{\infty} \equiv W_{\varepsilon}(E_N) \quad \text{on } E_N.$$

Therefore

$$\begin{aligned} |\hat{\mu}(0)| &\leq 1/W_{\varepsilon}(E_N) \int_{E_N} \left\{ \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right\} U_{\mu_N^*}(x) dx \\ &\leq 1/W_{\varepsilon}(E_N) \left\{ \hat{\mu}(0) + \sum_{k=0}^{N-1} \varepsilon_k \sum_{\nu=2^k}^{2^{k+1}-1} \hat{\mu}_N^*(\nu) \hat{\mu}(\nu) \right\}. \end{aligned}$$

By Schwarz's inequality,

$$\begin{aligned} |\hat{\mu}(0)|^2 &\leq 1/W_{\varepsilon}(E_N)^2 \left\{ |\hat{\mu}(0)|^2 + \sum_{k=0}^{\infty} \varepsilon_k \sum_{\nu=2^k}^{2^{k+1}-1} |\hat{\mu}(\nu)|^2 \right\} \\ &\quad \times \left\{ 1 + \sum_{k=0}^{N-1} \varepsilon_k \sum_{\nu=2^k}^{2^{k+1}-1} |\hat{\mu}_N^*(\nu)|^2 \right\}. \end{aligned}$$

Since

$$I_{\mu_N^*} = 1 + \sum_{n=0}^{\infty} \varepsilon_n \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}_N^*(k)|^2 = \|U_{\mu_N^*}\|_{\infty} = W_{\varepsilon}(E_N)$$

and $\mu \in \mathcal{U}_{\varepsilon}^+$, we have $|\hat{\mu}(0)|^2 \leq \text{Constant} \cdot 1/W_{\varepsilon}(E_N)$. By Lemma 5 and the hypothesis, we have $\lim_{N \rightarrow \infty} W_{\varepsilon}(E_N) = \infty$. Consequently we get $\hat{\mu}(0) = 0$.

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