PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 1, January 1988

ONE PARAMETER GROUPS AND THE CHARACTERIZATION OF THE SINE FUNCTION

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(Communicated by John B. Conway)

ABSTRACT. A real C^∞ function on $\mathbf{R}^1$, all of whose derivatives and (suitable) antiderivatives are uniformly bounded, is of the form $a\sin(x - x_0)$. We give an abstract version of this theorem.

The characterization of the sine function given above is due to Roe (see [**1**]). Precisely stated, it says that if $\{f_k\}_{k=-\infty}^{\infty}$ is such that $f_k' = f_{k+1}$ on $\mathbf{R}^1$ and $\sup_{k\in\mathbf{Z}} \|f_k\|_\infty < \infty$, then $f_0(x) = a\sin(x - x_0)$. Here is an abstract version.

THEOREM. *Let S_t be a strongly continuous, one-parameter group of linear operators on the Banach space X with* $\sup \|S_t\| = M < \infty$. *Let A be the infinitesimal generator of S_t with domain $\mathcal{D}$. Suppose $\{x_k\}_{-\infty}^{\infty}$ is a doubly infinite sequence of vectors in X such that each $x_k \in \mathcal{D}$, $Ax_k = x_{k+1}$ and $\sup_k \|x_k\| < \infty$. Then $x_0 = z_1 + z_2$ where $Az_1 = iz_1$, and $Az_2 = -iz_2$.*

PROOF. Assume $x_0 \neq 0$ and let Y be the linear space of all $\{y_k\}_{-\infty}^{\infty}$ satisfying the hypotheses of the Theorem (the algebraic operations are, of course, componentwise). Define $\|\{y_k\}\| = \sup_k \|y_k\|$. Then Y is a Banach space since $Y \subseteq l^\infty(X)$ and A is a closed operator.

Now define B to be the backward shift on Y, $B\{y_k\} = \{y_{k+1}\}$, so that B is an invertible isometry. Next note that $e^{tB}\{y_k\} = \{S_t y_k\}$. To see this, note that the kth coordinate of the left side is $\sum_{j=0}^{\infty}(t^j y_{k+j}/j!)$. On the other hand, $S_t y_k = \sum_{j=0}^{n}(t^j A^j y_k/j!) + R_n$ where

$$R_n = \frac{1}{(n+1)!}\int_0^t (t-u)^{n+1} S_u A^{n+1} y_k\, du.$$

Since $\|R_n\| \to 0$, equality follows. Hence $\|e^{tB}\| \leq M$ so that $\sigma(B)$ lies on the imaginary axis. But since B is an isometric isomorphism, $\sigma(B)$ lies on the unit circle. Thus $\sigma(B) \subseteq \{i, -i\}$.

Consequently, $Y = Y_i \oplus Y_{-i}$ where $Y_{\pm i}$ is invariant for B and $\sigma(B \mid Y_{\pm i}) = \pm i$. But on Y_i, $B = iI$. To see this, write $B = iI + Q$ where Q is quasinilpotent. A routine calculation shows that for complex z and $\varepsilon > 0$, $\|e^{zQ}\| \leq Ce^{\varepsilon|z|}$, and, of course, $\|e^{tQ}\| \leq M$, t real. So, for arbitrary $y \in Y_i$, $y^* \in Y_i^*$, the entire function $y^* e^{zQ} y$ is of exponential type zero and is bounded on the real axis and so is constant. Hence $Q = 0$. Similarly $B \mid Y_{-i} = -iI$ and the theorem follows.

To obtain Roe's result, let X be the Banach space of bounded, uniformly continuous functions on $\mathbf{R}$ with the sup norm and let S_t be left-translation by t. Then

Received by the editors August 1, 1986 and, in revised form, October 10, 1986.
1980 *Mathematics Subject Classification*. Primary 47D10, 47D05.

if $f \in \mathcal{D}$, $Af = f'$. If now f, f', and f'' are bounded and continuous, then Taylor's theorem with one or two terms shows that $f \in \mathcal{D}$ and our theorem gives $f(x) = ae^{ix} + be^{-ix}$.

Finally, and not unexpectedly, the Hilbert space case is even more transparent. For then, by a venerable theorem of Nagy (see [**2**, p. 132], iA is similar to a self-adjoint operator so we will assume this at the outset. Since each x_k is orthogonal to the kernel of A, we assume A is invertible. But then, if $iA = \int_{-\infty}^{\infty} \lambda \, dE_\lambda$, we get $\sup_k \int_{-\infty}^{\infty} \lambda^{2k} \, d\|E_\lambda x_0\|^2 < \infty$ and so $E(\cdot)x_0$ is supported on $\{-1, 1\}$.

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