INVARIANT SUBSPACES FOR DERIVATIONS

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ABSTRACT. In this article it is proved that most of the known sufficient conditions for a subspace from $\text{Lat } \mathcal{A}$ to be hyperinvariant are in fact also sufficient for this subspace to be invariant for all operators from $\text{Ad } \mathcal{A}$.

Let $\mathcal{A}$ be an algebra of operators on a Hilbert space H and let $\mathcal{A}'$ be its commutant. We write $\text{Lat } \mathcal{A}$ for the lattice of closed invariant subspaces of $\mathcal{A}$, and $\text{Hyp } \mathcal{A}$ for the subset of all hyperinvariant subspaces in $\text{Lat } \mathcal{A}$; i.e., all subspaces in $\text{Lat } \mathcal{A}$ invariant for all operators from $\mathcal{A}'$. For the weakly closed algebra $\mathcal{A} = \mathcal{A}(A)$ generated by a bounded operator A and the identity, Rosenthal [7] showed that if $M \in \text{Lat } \mathcal{A}$ is comparable to every point of $\text{Lat } \mathcal{A}$, then $M \in \text{Hyp } \mathcal{A}$. A related result due to Stampfli [8] says that if S is a countable subset of $\text{Lat } \mathcal{A}$ and if every element of S is comparable to every element of $\text{Lat } \mathcal{A} \setminus S$, then $S \subseteq \text{Hyp } \mathcal{A}$. Fillmore [3] extended this result and proved that if M and N from $\text{Lat } \mathcal{A}$ are comparable to every point of $\text{Lat } \mathcal{A}$ and the set $\{M, N\} = \{L \in \text{Lat } \mathcal{A} : M \subseteq L \subseteq N\}$ is countable, then $\{M, N\} \subseteq \text{Hyp } \mathcal{A}$.

In [1] Douglas and Pearcy initiated the study of topological properties of $\text{Lat } \mathcal{A}(A)$ considered as a metric space under the metric $d(M, N) = \|P_M - P_N\|$ where P_M and P_N denote the orthoprojections on M and N . They proved that if M is an inaccessible point of $\text{Lat } \mathcal{A}$, then $M \in \text{Hyp } \mathcal{A}$. In [2] they and Salinas showed that if S is a totally disconnected, open subset of $\text{Lat } \mathcal{A}$, then every subspace in S is hyperinvariant. From this the results of Rosenthal, Fillmore, and Stampfli follow immediately. In [2] a result of another kind was obtained which shows the existence of a hyperinvariant subspace if $\text{Lat } \mathcal{A}(A)$ contains two proper subspaces $M \subset N$ such that every subspace in $\text{Lat } \mathcal{A}(A)$ is comparable with either M or N . Murphy [6] proved that if $M, N \in \text{Hyp } \mathcal{A}$ and if $L \in \{M, N\}$ lies in no disc in $\{M, N\}$, then $L \in \text{Hyp } \mathcal{A}$.

In [4] for operator algebras $\mathcal{A}$, the Lie algebras $\text{Ad } \mathcal{A} = \{B \in B(H) : [B, A] = BA - AB \in \mathcal{A} \text{ for all } A \in \mathcal{A}\}$ were considered and the triplets $(\mathcal{A}, \text{Lat } \mathcal{A}, \text{Ad } \mathcal{A})$ were investigated. Operators from $\text{Ad } \mathcal{A}$ generate derivations on $\mathcal{A}$. It is easy to see that $\mathcal{A}$ and $\mathcal{A}'$ are Lie ideals in $\text{Ad } \mathcal{A}$. Although in many important cases (such as W^* - and CSL-algebras) $\text{Ad } \mathcal{A} = \mathcal{A} + \mathcal{A}'$, very often $\text{Ad } \mathcal{A}$ is much larger than $\mathcal{A} + \mathcal{A}'$ (see [4, 5]).

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By $\mathcal{L}(\mathcal{A})$ we denote the set of all subspaces in $\text{Lat } \mathcal{A}$ invariant for all operators from $\text{Ad } \mathcal{A}$. Obviously $\mathcal{L}(\mathcal{A}) \subseteq \text{Hyp } \mathcal{A}$ and every subspace in $\mathcal{L}(\mathcal{A})$ is invariant for all operators in the algebra generated by $\text{Ad } \mathcal{A}$.

In this article we investigate the sufficient conditions considered in [2, 3, 6, 7, 8] for a subspace or a set of subspaces from $\text{Lat } \mathcal{A}$ to belong to $\text{Hyp } \mathcal{A}$. We prove that these conditions are sufficient for them to belong to $\mathcal{L}(\mathcal{A})$. In Theorems 9 and 10 we also obtain some additional sufficient conditions for a set of subspaces from $\text{Lat } \mathcal{A}$ to belong to $\mathcal{L}(\mathcal{A})$. Theorem 11 shows that if S is an isolated subset of $\text{Lat } \mathcal{A}$, then the set S is invariant for $\text{Ad } \mathcal{A}$.

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1. Preliminary results. Let $B(H)$ be the algebra of all bounded operators on a Hilbert space H . If X is a set of closed subspaces of H , then by $\text{Alg } X$ we denote the algebra of all operators in $B(H)$ which leave every member of X invariant. For a bounded operator A and for a subspace M , AM denotes the linear subspace generated by $\{Ax : x \in M\}$ and M_A denotes $\exp(A)M$.

LEMMA 1 [1]. *If M, N are subspaces in H and $A, B \in B(H)$ are invertible, then $d(AM, BN) \leq \|A - B\|(\|A^{-1}\| + \|B^{-1}\|) + d(M, N)(\|A^{-1}\| \|B\| + \|B^{-1}\| \|A\|)$.*

Let $\mathbf{D}_r$ denote the disc of radius r : $\mathbf{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$. A disc of radius r in a topological space X is a subset of X homeomorphic to $\mathbf{D}_r$. Also define $\mathbf{D}_r(M, A)$ to be $\{M_{tA} : |t| \leq r\}$. Let $\sigma(A)$ denote the spectrum of the operator A .

LEMMA 2. *Let $A \in B(H)$, let M be a closed subspace of H and let $r < \ln 2/2\|A\|$. Then M_{tA} is a closed subspace for any $t \in \mathbb{C}$ and either $AM \subseteq M$ or $\mathbf{D}_r(M, A)$ is homeomorphic to $\mathbf{D}_r$.*

PROOF. M_{tA} is closed because $\exp(tA)$ is invertible for any $t \in \mathbb{C}$. By Lemma 1, for $t, z \in \mathbb{C}$

$$d(M_{tA}, M_{zA}) \leq \|\exp(tA) - \exp(zA)\| (\|\exp(-tA)\| + \|\exp(-zA)\|).$$

Therefore the mapping $t \rightarrow M_{tA}$ is continuous.

Suppose that $M_{tA} = M_{zA}$ for some $t, z \in \mathbf{D}_r$. Put $u = t - z$. Then $|u| \leq 2r$ and $M_{uA} = M$. Set $B = \exp(uA)$; then $BM = M$. Since $\|uA\| < \ln 2$,

$$\|B - I\| = \left\| \sum_{n=1}^{\infty} \frac{(uA)^n}{n!} \right\| \leq \sum_{n=1}^{\infty} \frac{\|uA\|^n}{n!} = e^{\|uA\|} - 1 < 1,$$

and $\sigma(B) = \exp(\sigma(uA))$. Therefore $|\lambda - 1| < 1$ for any $\lambda \in \sigma(B)$ and

$$(1) \quad uA = \ln B = \sum_{n=1}^{\infty} (-1)^{n-1} (B - I)^n / n.$$

It follows from (1) that $AM \subseteq M$. Thus if M is not invariant for A , then $t \rightarrow M_{tA}$ is a continuous and one-to-one mapping of $\mathbf{D}_r$ onto $\mathbf{D}_r(M, A)$. Since $\mathbf{D}_r$ is a compact subset of $\mathbb{C}$, $\mathbf{D}_r$ is homeomorphic to $\mathbf{D}_r(M, A)$ and the lemma is proved.

LEMMA 3 [4]. *Let $A, B \in B(H)$.*

- (i) *If A is invertible, then $A \exp(B) A^{-1} = \exp(ABA^{-1})$.*
- (ii) *Let $\text{ad } B(A) = [B, A]$. Then $\exp(B) A \exp(-B) = \exp(\text{ad } B)(A)$.*

LEMMA 4. (i) Let $\mathcal{A}$ be an operator algebra and let $B \in \text{Ad } \mathcal{A}$. The transformation $M \in \text{Lat } \mathcal{A} \rightarrow M_B$ is a homeomorphism of $\text{Lat } \mathcal{A}$ onto itself.

(ii) Let X be a set of closed subspaces of H and let $B \in B(H)$. If $M_{tB} \in X$ for every $M \in X$ and for every $t \in \mathbb{C}$, then $B \in \text{Ad}(\text{Alg } X)$.

PROOF. Let $x \in M$ and $A \in \mathcal{A}$. By Lemma 3,

$$\begin{aligned} y &= \exp(-B)A \exp(B)x = \exp(-\text{ad } B)(A)x \\ &= \sum_{k=0}^{\infty} (-1)^k (\text{ad } B)^k(A)x/k!. \end{aligned}$$

Since $B \in \text{Ad } \mathcal{A}$, for every k the operator $(\text{ad } B)^k(A)$ belongs to $\mathcal{A}$ and therefore every $(\text{ad } B)^k(A)x$ belongs to M . Since M is closed, y belongs to M . Therefore

$$A \exp(B)x = \exp(B)y \in M_B,$$

so that $M_B \in \text{Lat } \mathcal{A}$. Since $\exp(B)$ is invertible, $M \rightarrow M_B$ is a one-to-one mapping of $\text{Lat } \mathcal{A}$ onto itself. It follows from Lemma 1 that the mapping is continuous and the inverse mapping is also continuous, so that the mapping is a homeomorphism and (i) is proved.

For every $A \in \text{Alg } X$, for every $M \in X$, and for every $t \in \mathbb{C}$,

$$\exp(-tB)A \exp(tB)M = \exp(-tB)AM_{tB} \subseteq \exp(-tB)M_{tB} = M,$$

since $M_{tB} \in X$. Therefore $\exp(-tB)A \exp(tB) \in \text{Alg } X$. $\text{Alg } X$ is closed in the norm topology. Hence

$$\frac{d}{dt} [\exp(-tB)A \exp(tB)] \big|_{t=0} = -[B, A] \in \text{Alg } X,$$

so that $B \in \text{Ad}(\text{Alg } X)$, which completes the proof of the lemma.

2. Invariant and discrete subsets in $\text{Lat } \mathcal{A}$. We say that a subset X of $\text{Lat } \mathcal{A}$ is invariant for $\text{Ad } \mathcal{A}$ if for every $M \in X$ and for every $B \in \text{Ad } \mathcal{A}$, we have that $M_B \in X$.

From Lemma 4 we immediately obtain the following theorem.

THEOREM 5 [4]. If X is a subset of $\text{Lat } \mathcal{A}$ invariant for $\text{Ad } \mathcal{A}$, then $\text{Ad } \mathcal{A} \subseteq \text{Ad}(\text{Alg } X)$. In particular $\text{Ad } \mathcal{A} \subseteq \text{Ad}(\text{Alg Lat } \mathcal{A})$.

THEOREM 6. Let $M, N \in \mathcal{L}(\mathcal{A})$ and let $M \subset N$. If $L \in \{M, N\}$ lies in no disc in $\{M, N\}$, then $L \in \mathcal{L}(\mathcal{A})$.

PROOF. For any $L \in \{M, N\}$ and for $B \in \text{Ad } \mathcal{A}$, $L_B \in \text{Lat } \mathcal{A}$ and

$$M = M_B \subseteq L_B \subseteq N_B = N.$$

Therefore $\{M, N\}$ is invariant for $\text{Ad } \mathcal{A}$, so that $\mathbf{D}_r(L, B) \subseteq \{M, N\}$ for any positive r . If L lies in no disc in $\{M, N\}$, then it follows from Lemma 2 that $BL \subseteq L$ for any $B \in \text{Ad } \mathcal{A}$ and the theorem is proved.

We say that a point $L \in \{M, N\}$ is *inaccessible* in $\{M, N\}$ if its path-component in $\{M, N\}$ is just $\{L\}$.

COROLLARY 7. *Let M and N belong to $\mathcal{L}(\mathcal{A})$ and let $M \subset N$. If L is isolated or inaccessible in $\{M, N\}$, then $L \in \mathcal{L}(\mathcal{A})$. In particular, if L is isolated or inaccessible in $\text{Lat } \mathcal{A}$, then $L \in \mathcal{L}(\mathcal{A})$.*

REMARK. In [1] it was proved that $L \in \text{Hyp } \mathcal{A}$. It follows from Corollary 7 that in fact L belongs to $\mathcal{L}(\mathcal{A})$.

THEOREM 8. *Suppose that S is a discrete subset of $\text{Lat } \mathcal{A}$ such that the orthoprojection on every subspace in S commutes with the orthoprojection on each subspace in the complement of S . Then every subspace in S belongs to $\mathcal{L}(\mathcal{A})$.*

PROOF. If M and N are two subspaces of H such that P_M commutes with P_N , then $d(M, N) = 1$. Thus $d(M, N) = 1$ for every M in S and every N in the complement of S . Since S is discrete, every subspace in S is isolated and the desired conclusion follows from Corollary 7.

REMARK. In [2, 3, 7, 8] it was proved that $S \subseteq \text{Hyp } \mathcal{A}$. From Theorem 8 it follows that in fact $S \subseteq \mathcal{L}(\mathcal{A})$.

The condition on S can be made even weaker than in Theorem 8. Let P_M be the orthoprojection on a subspace M and let Q_M be the projection on M along a subspace L . It is easy to see that

$$(2) \quad Q_M P_M = P_M \quad \text{and} \quad P_M Q_M = Q_M.$$

THEOREM 9. *Suppose that S is a discrete subset in $\text{Lat } \mathcal{A}$. If for every M in S there exists a projection Q_M such that every $N \in \text{Lat } \mathcal{A} \setminus S$ is invariant for Q_M , then $S \subseteq \mathcal{L}(\mathcal{A})$.*

PROOF. Let Q_M be the projection on $M \in S$ which satisfies the condition of the theorem and let $B \in \text{Ad } \mathcal{A}$. By Lemma 1

$$(3) \quad \begin{aligned} d(M_{tB}, M) &\leq \|\exp(tB) - I\| (\|\exp(-tB)\| + 1) \\ &\leq (e^{\|tB\|} - 1)(e^{\|tB\|} + 1). \end{aligned}$$

Since S is discrete, there exists a positive r such that $M_{tB} \notin S \setminus M$ for all $|t| \leq r$. By Lemma 4(i), $M_{tB} \in \text{Lat } \mathcal{A}$ and therefore $M_{tB} \in (\text{Lat } \mathcal{A} \setminus S) \cup M$. Hence $Q_M M_{tB} \subseteq M_{tB}$. From this it follows that for every $x \in M$

$$x(t) = \exp(-tB)Q_M \exp(tB)x \in M.$$

Differentiating at $t = 0$ and taking into account that M is closed, we obtain

$$\left. \frac{dx(t)}{dt} \right|_{t=0} = [Q_M, B]x \in M,$$

so that M is invariant for $[Q_M, B]$. Let P_M be the orthoprojection on M . Then $P_M [Q_M, B] P_M = [Q_M, B] P_M$. Using (2) we obtain

$$\begin{aligned} P_M [Q_M, B] P_M &= P_M Q_M B P_M - P_M B Q_M P_M = Q_M B P_M - P_M B P_M, \\ [Q_M, B] P_M &= Q_M B P_M - B Q_M P_M = Q_M B P_M - B P_M. \end{aligned}$$

Therefore $P_M B P_M = B P_M$, so that $M \in \mathcal{L}(\mathcal{A})$. The theorem is proved.

Combining Theorems 5 and 9 we obtain the following general theorem.

THEOREM 10. *Let X be a subset in $\text{Lat } \mathcal{A}$ invariant for $\text{Ad } \mathcal{A}$ and let S be a discrete subset in X such that for every $M \in S$ there exists a projection Q_M on M such that every $N \in X \setminus S$ is invariant for Q_M . Then $S \subseteq \mathcal{L}(\mathcal{A})$.*

PROOF. Let $\mathcal{B} = \text{Alg } X$. Then $\text{Lat } \mathcal{B} = X$ and, by Theorem 9, every M in S is invariant for $\text{Ad } \mathcal{B}$. By Theorem 5, $\text{Ad } \mathcal{A} \subseteq \text{Ad } \mathcal{B}$, which completes the proof.

Now we shall consider a sufficient metric condition for a subset of $\text{Lat } \mathcal{A}$ to be invariant for $\text{Ad } \mathcal{A}$.

THEOREM 11. *Let S be a subset of $\text{Lat } \mathcal{A}$ and let $m = \inf\{d(M, N) : M \in S, N \in \text{Lat } \mathcal{A} \setminus S\}$. If $m \neq 0$, then S is invariant for $\text{Ad } \mathcal{A}$.*

PROOF. Let $B \in \text{Ad } \mathcal{A}$. By (3),

$$d(M_{tB}, M) \leq (e^{2\|tB\|} - 1)$$

and we can choose a positive r such that $d(M_{tB}, M) \leq m/2$ for all $|t| \leq r$ and for all $M \in S$. From this it follows that $M_{tB} \in S$. For any complex λ , choosing an integer n so that $|\lambda/n| \leq r$ we obtain

$$M_{\lambda B} = \exp(\lambda B)M = \exp(\lambda B/n) \cdots \exp(\lambda B/n)M \in S.$$

Since λ and B are arbitrary, this completes the proof.

Let Ω be an open subset of $\text{Lat } \mathcal{A}$ and let Γ be an arcwise connected component of $\text{Lat } \mathcal{A}$. Set

$$\mathcal{T} = \bigcap_{M \in \Omega \cap \Gamma} M \quad \text{and} \quad \mathcal{K} = \bigvee_{M \in \Omega \cap \Gamma} M.$$

In [2, Theorem 2.2] it was proved that $\mathcal{T}$ and $\mathcal{K}$ belonged to $\text{Hyp } \mathcal{A}$. Repeating, essentially the same argument as in [2], we obtain the following theorem.

THEOREM 12. *$\mathcal{T}$ and $\mathcal{K}$ belong to $\mathcal{L}(\mathcal{A})$.*

PROOF. In [2] it was shown that it is sufficient to prove that

$$B\mathcal{T} \subseteq M \quad \text{and} \quad BM \subseteq \mathcal{K}$$

for every $M \in \Omega \cap \Gamma$ and for every $B \in \text{Ad } \mathcal{A}$. Since Ω is open, repeating the same argument as in Theorem 11, we obtain that there exists a positive r such that $M_{tB} \in \Omega \cap \Gamma$ for all $|t| < r$. Let $x \in \mathcal{T}$. Then $x \in \bigcap_{|t| < r} M_{tB}$. Therefore for $|\lambda| < r$

$$x(\lambda) = \exp(\lambda B)x \in \bigcap_{|t| < r} M_{(t+\lambda)B} \subseteq M.$$

Differentiating $x(\lambda)$ at $\lambda = 0$ and taking into account that M is closed, we obtain that $Bx \in M$. Therefore $B\mathcal{T} \subseteq M$.

If $x \in M$, then $x(t) = \exp(tB)x \in M_{tB} \subseteq \mathcal{K}$ for all $|t| < r$. Since $\mathcal{K}$ is closed, differentiating $x(t)$ at $t = 0$, we obtain that $Bx \in \mathcal{K}$ which completes the proof.

Repeating the argument of [2, Theorem 3.1] and using Theorem 12 we immediately obtain the following theorem.

THEOREM 13. *Let $\mathcal{A}$ be an operator algebra, and suppose that there exist two nonzero proper subspaces M and N in $\text{Lat } \mathcal{A}$ such that $M \subset N$ and such that*

every subspace in $\text{Lat } \mathcal{A}$ is comparable with either M or N . Then $\mathcal{L}(\mathcal{A})$ contains a nontrivial subspace.

THEOREM 14. *Let $Q_M \in \text{Ad } \mathcal{A}$ be the projection on a subspace M along a subspace L . If there exists a subspace N in $\text{Lat } \mathcal{A}$ complementary to M , then L belongs to $\text{Lat } \mathcal{A}$. If, in addition, N belongs to $\mathcal{L}(\mathcal{A})$, then $L \in \mathcal{L}(\mathcal{A})$.*

PROOF. If the orthoprojection P_M on a subspace M belongs to $\mathcal{A}'$, then M and $M^\perp$ belong to $\text{Lat } \mathcal{A}$. If the projection Q_M on M along a subspace L belongs to $\mathcal{A}'$, then it is obvious that M and L belong to $\text{Lat } \mathcal{A}$. Suppose now that the projection $Q_M \in \text{Ad } \mathcal{A}$. It is easy to see that there exists a bounded operator T from $M^\perp$ into M such that $Q_M = \begin{pmatrix} I & T \\ 0 & 0 \end{pmatrix}$. Let $A = (A_{ij}) \in \mathcal{A}$ for $1 \leq i, j \leq 2$. Then $[Q_M, A]$ and $[Q_M, [Q_M, A]]$ belong to $\mathcal{A}$ and it is easy to check that

$$A_1 = [Q_M, A] + [Q_M, [Q_M, A]] = \begin{pmatrix} 0 & B_{12} \\ 0 & 0 \end{pmatrix} \in \mathcal{A},$$

where $B_{12} = 2(A_{12} + TA_{22} - A_{11}T - TA_{21}T)$.

Assume now that there exists a subspace $N \in \text{Lat } \mathcal{A}$ complementary to M , that is, $M \dot{+} N = H$. It is easy to show that there exists a bounded operator S from $M^\perp$ into M such that $N = \{ \begin{pmatrix} Sy \\ y \end{pmatrix} : y \in M^\perp \}$. Then $A_1 N \subseteq N$, so that $\begin{pmatrix} B_{12}y \\ 0 \end{pmatrix} \in N$ for all $y \in M^\perp$. Therefore

$$(4) \quad B_{12} = 2(A_{12} + TA_{22} - A_{11}T - TA_{21}T) = 0.$$

The projection $I - Q_M = \begin{pmatrix} 0 & -T \\ 0 & I \end{pmatrix}$ belongs to $\text{Ad } \mathcal{A}$ and is the projection on the subspace $L = \{ \begin{pmatrix} -Ty \\ y \end{pmatrix} : y \in M^\perp \}$ along M . For every $A \in \mathcal{A}$ and for every $z = \begin{pmatrix} -Ty \\ y \end{pmatrix} \in L$, we have that $Az = \begin{pmatrix} x \\ y_1 \end{pmatrix}$, where $y_1 = (A_{22} - A_{21}T)y$ and $x = (A_{12} - A_{11}T)y$. It follows from (4) that $x = -Ty_1$, so that $L \in \text{Lat } \mathcal{A}$. If $N \in \mathcal{L}(\mathcal{A})$, then considering A from $\text{Ad } \mathcal{A}$ and using the same argument, we obtain that $L \in \mathcal{L}(\mathcal{A})$ which completes the proof.

COROLLARY 15. *Let an orthoprojection P_M on M belong to $\text{Ad } \mathcal{A}$. If there exists a reducing subspace N in $\text{Lat } \mathcal{A}$ such that M and N are complementary ($d(M, N^\perp) < 1$), then M is a reducing subspace in $\text{Lat } \mathcal{A}$.*

PROOF. By Theorem 14, $M^\perp \in \text{Lat } \mathcal{A}$. Since N is a reducing subspace in $\text{Lat } \mathcal{A}$, N and $N^\perp$ belong to $\text{Lat } \mathcal{A}$. In [1] it was proved that M and N are complementary if

$$d(M, N^\perp) = \|P_M + P_N - I\| < 1.$$

We have that $P_{M^\perp} = I - P_M \in \text{Ad } \mathcal{A}$ and that

$$d(M^\perp, N) = \|I - P_M - P_N\| = d(M, N^\perp) < 1,$$

so that $M^\perp$ and $N^\perp$ are also complementary. From Theorem 14 it follows that M belongs to $\text{Lat } \mathcal{A}$, so that M is a reducing subspace in $\text{Lat } \mathcal{A}$.

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