RANDOM SIGN EMBEDDINGS FROM l_r^n , $2 < r < \infty$

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ABSTRACT. Estimates for any ideal norm of a “random sign embedding” from l_r^n into l_r^m , $2 < r < \infty$, are given in terms of the corresponding ideal norm of the identity of l_r^k , $k = k(n, m, r)$.

1. Introduction. A norm one linear operator u from l_r^n ($1 \leq r < \infty$) into a Banach space is called a δ -sign embedding (where $\delta > 0$) provided

$$\left\| u \sum_{i=1}^n \varepsilon_i e_i \right\| \geq \delta n^{1/r}$$

for all choices of signs $\varepsilon_i \pm 1$. (Here $(e_i)_{i=1}^n$ denotes the unit vector basis for l_r^n .) If the weaker condition

$$\text{Average} \left\{ \left\| u \sum_{i=1}^n \varepsilon_i e_i \right\| : (\varepsilon_i) \in \{-1, 1\}^n \right\} \geq \delta n^{1/r}$$

is satisfied, we say that u is a *random δ -sign embedding*.

The most striking result concerning these concepts is J. Elton's theorem [E] (extended by A. Pajor [Pa] to complex Banach spaces) that if u is a random δ -sign embedding from l_1^n , then there is a subset A of $\{1, \dots, n\}$ with cardinality $k \equiv |A| \geq \varepsilon n$ so that the restriction of u to l_1^A has isomorphism constant at most K , where $\varepsilon > 0$ and $K < \infty$ are constants which depend only on δ . Predating Elton's theorem was the result from [JS] that when the random δ -sign embedding u from l_1^n takes values in L_1 , the space ul_1^A is a well-complemented copy of l_1^k in L_1 and hence the identity operator on l_1^k well factors through the operator u . From results stated but not proved in [JS] it can be derived that if u is a random δ -sign embedding from l_p^n into L_p for $1 < p < 2$, then the identity on l_p^k well factors through u for some $k \geq n^{1-\varepsilon}$ and any $\varepsilon > 0$ (where, of course, how well depends on ε). Actually, simpler considerations yield that the identity on l_p^k with k proportional to n weakly factorizes through such a u ; this fact is recorded in Remark 3. (“Weak factorization” is defined in Proposition 2.)

There cannot be any such results in the range $r > 2$ because

$$n^{1/r-1/2} I: l_r^n \rightarrow l_2^n$$

is a 1-sign embedding and hence by the results of [FLM or BDGJN], there is for some $\delta > 0$ a δ -sign embedding u from l_r^n into l_r^m with $m \leq n^{r/2}$ whose factorization

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constant $\gamma_2(u)$ through a Hilbert space is less than 2. The purpose of this note is to observe that this phenomenon cannot occur if m is proportional to n ; for example, if $2 < r < \infty$ and $u: l_r^n \rightarrow l_r^{2n}$ is a random δ -sign embedding, then the identity operator on l_r^k with k proportional to n has a good weak factorization through u , which yields that for any ideal norm α ,

$$\alpha(u) \geq \varepsilon \alpha(l_r^n),$$

where $\varepsilon > 0$ is a constant which depends only on δ and r and

$$\alpha(l_r^n) \equiv \alpha(I: l_r^n \rightarrow l_r^n).$$

We are interested in this result mainly for the ideal norm S_p , $1 \leq p < \infty$, defined by

$$S_p(u: X \rightarrow Y) = \inf \{ \|w\| \|v\| : u = vw, w: X \rightarrow E, v: E \rightarrow Y, \\ E \text{ a subspace of } L_p(\mu) \text{ for some measure } \mu \}.$$

This case is applied in [CJ]; indeed, the possibility of such an application motivated this note. In fact, in the first version of this note, Theorem 1 was stated only for $\alpha = S_p$. We are indebted to the referee for pointing out that our argument yields Proposition 2 and the present statement of Theorem 1.

In §2 we repeat the known observation that, up to constants depending on p and r , if $r > 2$ and $1 < p < \infty$, then $S_p(l_r^n)$ is the same as $\gamma_p(l_r^n)$, the factorization constant of the identity of l_r^n through an L_p space. This latter constant is known, up to constants depending on r and p , to be of order n^α for an appropriate exponent $\alpha = \alpha(p, r)$. The values for α are catalogued in Pietsch's book [P] and repeated in §2.

We use standard Banach space theory notation and terminology as may be found in [LT].

2. Random sign embeddings from l_r^n . We now state the result of this note.

THEOREM 1. *Let $2 < r < \infty$. If $u: l_r^n \rightarrow l_r^m$ is a random δ -sign embedding, then for any ideal norm α ,*

$$\alpha(u) \geq \tau \alpha(l_r^k),$$

where

$$\tau = 2^{-1/r} (\delta/r^{1/2})^{2(r-1)/(r-2)} (n/m)^{1/(r-2)}$$

and

$$k \geq \tau^r n / (1 - \tau^r).$$

As pointed out by the referee, Theorem 1 is an immediate consequence of Proposition 2. The referee's concept of weak factorization introduced in the proposition is the obvious operator theoretic analogue to the weak distance studied by N. Tomczak-Jaegermann [T-J].

PROPOSITION 2. *Let r, u, δ, τ and k be as in Theorem 1. Then $I: l_r^k \rightarrow l_r^k$ τ^{-1} -weakly factorizes through u ; i.e., I is in the convex hull of the set*

$$\{wuv: v: l_r^k \rightarrow l_r^n; w: l_r^m \rightarrow l_r^k; \|w\| \cdot \|v\| \leq \tau^{-1}\}.$$

PROOF. From Khintchine's inequality we have

$$(i) \quad \left\| \left(\sum_{i=1}^n |ue_i|^2 \right)^{1/2} \right\|_r \geq \delta r^{-1/2} n^{1/r},$$

where the functional operations $|x|^2$ and $|x|^{1/2}$ are taken with respect to the lattice structure of l_r^m .

On the other hand, regarding u as an operator into l_∞^m ,

$$\|u: l_r^n \rightarrow l_\infty^m\| \leq \|u: l_r^n \rightarrow l_r^m\| = 1,$$

so, setting $1/r + 1/s = 1$, we easily check that

$$(ii) \quad \left\| \left(\sum_{i=1}^n |ue_i|^s \right)^{1/s} \right\|_\infty \leq 1.$$

Indeed,

$$\begin{aligned} \left\| \left(\sum_{i=1}^n |ue_i|^s \right)^{1/s} \right\|_\infty &= \max_{1 \leq j \leq m} \sup \left\{ \sum_{i=1}^n a_i ue_i(j) : \sum_{i=1}^n |a_i|^r = 1 \right\} \\ &= \sup \left\{ \left\| u \sum_{i=1}^n a_i e_i \right\|_\infty : \sum_{i=1}^n |a_i|^r = 1 \right\} \\ &= \|u: l_r^n \rightarrow l_\infty^m\| \leq 1. \end{aligned}$$

An extrapolation argument now yields

$$(iii) \quad \left\| \max_{1 \leq i \leq n} |ue_i| \right\|_r \geq 2^{1/r} \tau n^{1/r}.$$

To see this, set $\theta = s/2$, so that $1/2 = \theta/s + (1 - \theta)/\infty$. Then

$$\begin{aligned} \delta r^{-1/2} n^{1/r} &\leq \left\| \left(\sum_{i=1}^n |ue_i|^2 \right)^{1/2} \right\|_r \quad (\text{by (i)}) \\ &\leq \left\| \left(\sum_{i=1}^n |ue_i|^s \right)^{\theta/s} \max_{1 \leq i \leq n} |ue_i|^{1-\theta} \right\|_r \\ &\leq \left\| \left(\sum_{i=1}^n |ue_i|^s \right)^{1/s} \right\|_r^\theta \left\| \max_{1 \leq i \leq n} |ue_i| \right\|_r^{1-\theta} \\ &\leq m^{\theta/r} \left\| \left(\sum_{i=1}^n |ue_i|^s \right)^{1/s} \right\|_\infty^\theta \left\| \max_{1 \leq i \leq n} |ue_i| \right\|_r^{1-\theta} \\ &\leq m^{\theta/r} \left\| \max_{1 \leq i \leq n} |ue_i| \right\|_r^{1-\theta} \quad (\text{by (ii)}). \end{aligned}$$

The conclusion of Proposition 2 follows in a formal way from condition (iii) via a combination of more-or-less standard techniques.

For $1 \leq i \leq n$, define

$$A'_i = \left\{ j : |ue_i(j)| = \max_{1 \leq k \leq m} |ue_i(k)| \right\}$$

and set $A_i = A'_i \setminus \bigcup_{j < i} A'_j$. Since the A_i 's are pairwise disjoint, we have

$$(iv) \quad \left(\sum_{i=1}^n \|ue_i 1_{A_i}\|_r^r \right)^{1/r} = \left\| \sum_{i=1}^n ue_i 1_{A_i} \right\|_r = \left\| \max_{1 \leq i \leq r} |ue_i| \right\|_r.$$

Setting $A = \{i \leq n : \|ue_i 1_{A_i}\| \geq \tau\}$, we check that

$$(v) \quad |A| \equiv k \geq \tau^r n / (1 - \tau^r).$$

Indeed, since $\|ue_i 1_{A_i}\|_r \leq 1$ for each i , (iv) and (iii) yield $k + \tau^r(n - k) \geq 2\tau^r n$ so that $(1 - \tau^r)k \geq \tau^r n$.

The rest of the proof is really just an application of Tong's diagonal principle (see [LT, p. 20]).

Assume, for notational convenience, that $A = \{1, \dots, k\}$ and let P be the norm one projection from l_r^m onto $\text{span}(ue_i 1_{A_i})_{i=1}^k$. For each ε in $\{-1, 1\}^k$, define $v_\varepsilon: l_r^k \rightarrow l_r^n$ and $\tilde{w}_\varepsilon: \text{span}(ue_i 1_{A_i})_{i=1}^k \rightarrow l_r^k$ by setting

$$v_\varepsilon e_i = \varepsilon(i) e_i; \quad \tilde{w}_\varepsilon(ue_i 1_{A_i}) = \varepsilon(i) e_i$$

and let $w_\varepsilon = \tilde{w}_\varepsilon P$; $w_\varepsilon: l_r^m \rightarrow l_r^k$. Evidently $\|v_\varepsilon\| = 1$, $\|w_\varepsilon\| \leq \tau^{-1}$, and it is easy to check that $I: l_r^k \rightarrow l_r^k = \text{Average}_\varepsilon w_\varepsilon u v_\varepsilon$. $\square$

REMARK 3. Suppose that $u: l_r^k \rightarrow L_r$ is a random δ -sign embedding and $1 < r < 2$. Khintchine's inequality yields that

$$(\delta/2)n^{1/r} \leq \left\| \left(\sum_{i=1}^n |ue_i|^2 \right)^{1/2} \right\| \leq \left\| \left(\sum_{i=1}^n |ue_i|^r \right)^{1/r} \right\| \leq n^{1/r},$$

so the extrapolation argument used in the proof of Proposition 2 gives

$$(\delta/2)^{2/(2-r)} n^{1/r} \leq \left\| \max_{1 \leq i \leq n} |ue_i| \right\|.$$

The further argument in Proposition 2 shows that the identity operator on l_r^k with k proportional to n has a good weak factorization through u .

We close with a catalogue of estimates for $S_p(l_r^k)$ when $2 < r < \infty$. First, a simple exchangeability argument (see, for example, [JMST, p. 34]) shows that $S_p(l_r^k)$ is obtained via an operator which maps $(e_i)_{i=1}^k$ onto a normalized 1-symmetric basic sequence in L_p . Now for $1 \leq p \leq 2$, L_p has cotype 2, so

$$S_p(l_r^k) \sim d(l_r^k, l_2^k) = k^{1/2-1/r}$$

and the constant of equivalence is absolute.

For $2 < p < r$, L_p has cotype p with constant 1 if the average in the definition of cotype is in the l_p -sense, so in this range

$$S_p(l_r^k) = d(l_r^k, l_p^k) = k^{1/p-1/r}.$$

When $2 < p < \infty$, up to a constant depending on p , all the normalized 1-symmetric sequences are by Theorem 1.1 of [JMST] just symmetric versions of

Rosenthal's X_p basis [R], and all X_p spaces are K_p -isomorphic to K_p -complemented subspaces of L_p . Consequently, for any $1 \leq r \leq \infty$, $S_p(l_r^k) \sim \gamma_p(l_r^k)$ up to a constant depending only on p . It is possible, but not completely straightforward, to compute $S_p(l_r^k)$ or $\gamma_p(l_r^k)$ up to constants depending on p when $2 < r < p$ by computing the norm of the identity operators between l_r^k and symmetric X_p^k spaces; however, these parameters were first calculated in a different way by Gluskin, Pietsch, and Puhl [GPP, Pi]. They checked that in the range $2 < r < p < \infty$, up to constants depending on p , we have

$$\gamma_p(l_r^k) \sim n^\alpha, \quad \text{where } \alpha = (1/r - 1/p)(1/2 - 1/r)/(1/2 - 1/p).$$

REMARK 4. After the research on this paper was completed, J. Bourgain and L. Tzafriri proved the nice result that condition (iii) in the proof of Proposition 2 implies that the identity operator on l_r^k $K(\tau)$ -factors through u for some $k \geq \varepsilon(\tau)n$.

REFERENCES

- [BDGJN] G. Bennet, L. E. Dor, V. Goodman, W. B. Johnson, and C. M. Newman, *On uncomplemented subspaces of L^p , $1 < p < 2$* , Israel J. Math. **26** (1977), 178–187.
- [CJ] C. M. Cho and W. B. Johnson, *M-ideals and ideals in $L(X)$* , J. Operator Theory **16** (1986), 245–260.
- [E] J. Elton, *Sign-embedding on l_1^n* , Trans. Amer. Math. Soc. **279** (1983), 113–124.
- [FLM] T. Figiel, J. Lindenstrauss, and V. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. **139** (1977), 53–94.
- [GPP] E. D. Gluskin, A. Pietsch, and J. Puhl, *A generalization of Khintchine's inequality and its application in the theory of operator ideals*, Studia Math. **68** (1980), 149–155.
- [JMST] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc., no. 217, vol. 19 (1979).
- [JS] W. B. Johnson and G. Schechtman, *On subspaces of L_1 with maximal distances to Euclidean space*, Proc. Research Workshop on Banach space Theory, Univ. of Iowa, 1981, pp. 83–96.
- [LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I, Sequence spaces*, Springer-Verlag, Berlin and New York, 1977.
- [Pa] A. Pajor, *Prolongement de l_1^n dans les espaces de Banach complexes*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), 741–743.
- [P] A. Pietsch, *Operator ideals*, North-Holland, Amsterdam, 1980.
- [Pi] —, *Rosenthal's inequality and its application in the theory of operator ideals*, Proc. Internat. Conf. Operator Algebras, Ideals, ..., Teubner-Texte zur Math., Leipzig, 1978, pp. 80–88.
- [R] H. P. Rosenthal, *On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables*, Israel J. Math. **8** (1970), 273–303.
- [T-J] N. Tomczak-Jaegermann, *The weak distance between finite dimensional Banach spaces*, Math. Nachr. **119** (1984), 291–307.

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