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ON IMMERSED COMPACT SUBMANIFOLDS OF EUCLIDEAN SPACE

MARCO RIGOLI

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ABSTRACT. Given an immersion $f\colon M \to \mathbf{R}^n$ of a compact Riemannian manifold M we prove a simple criterion involving the tension field of f to determine whether or not f is an isometry.

1. Introduction. Let $f\colon M \to \mathbf{R}^n$ be an immersion of a Riemannian manifold into Euclidean space. A natural problem is to determine whether or not f is an isometry. In this note we give a proof of the following simple result (see §2 for details).

THEOREM. *Let M be an m-dimensional, compact, oriented, Riemannian manifold with metric ds^2 and let $f\colon M \to \mathbf{R}^n$ be an immersion. Set $d\sigma^2$ for the induced metric on M via f, u for the ratio of the volume elements, τ for the tension field of f and H for the mean curvature vector of $f\colon (M, d\sigma^2) \to \mathbf{R}^n$. Then f is an isometry iff*

(i) $\langle f, \tau - umH\rangle \geq 0$ *and*
(j) f *is volume decreasing for* $m \geq 3$,
(jj) f *is volume preserving for* $m = 2$,
(jjj) f *is volume increasing for* $m = 1$.

REMARKS. 1. The necessity of the above conditions is clear. Indeed if f is an isometry then $u \equiv 1$, that is f is volume preserving, and $\tau = mH$ (see §2).

2. For $m = 2$ in the proof of the theorem it will become apparent that (i) alone implies that f is conformal. We wish to state this in the form of the following:

PROPOSITION. *Let $f\colon M \to \mathbf{R}^n$ be an immersed compact Riemannian surface. Then f is conformal iff $\tau = 2uH$.*

PROOF. Sufficiency follows from above. Necessity follows from a well-known computation of the tension field (for instance see Hoffman-Osserman [**1**]).

3. A step in the proof of the theorem is based on the following result from linear algebra. Let V be a real m-dimensional vector space, G an inner product in V and H a symmetric semi-positive-definite bilinear form. Let (g_{ij}), (h_{ij}) be their matrices with respect to a basis of V. Set $g = \det(g_{ij})$ and $h = \det(h_{ij})$; clearly $g > 0$ and $h \geq 0$. For λ a parameter consider the determinant

$$\det(g_{ij} + \lambda h_{ij}) = g + mP\lambda + \cdots + h\lambda^m,$$

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where P is a polynomial in the entries of the matrices of G and H. It is easily verified that the quantity P/g is independent of the basis chosen in V; we claim that

$$P/g \geq (h/g)^{1/m} \tag{1}$$

where the equality sign holds iff $h_{ij} = \rho g_{ij}$ for a certain ρ. Indeed we choose a basis of V such that $g_{ij} = \delta_{ij}$ and (h_{ij}) is diagonal so that $(h_{ij}) = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$. Then (1) becomes

$$\frac{1}{m}\sum_{i=1}^{m} \lambda_i \geq \left(\prod_{i=1}^{m} \lambda_i\right)^{1/m} \tag{2}$$

and the result is known as a standard inequality.

2. Preliminaries on differential geometry. We realize the Euclidean space $\mathbf{R}^n$ as the homogeneous space $E(n)/\mathrm{SO}(n)$, where $E(n) = \mathrm{SO}(n) \times \mathbf{R}^n$ is the group of rigid motions and $\mathrm{SO}(n)$ its isotropy subgroup at the origin 0 of $\mathbf{R}^n$. From now on we fix the indices convention $1 \leq A, B, \dots \leq n$, $1 \leq i, j, \dots \leq m$, $m + 1 \leq \alpha$, $\beta, \dots \leq n$. If Θ^A_B, Θ^A denote the components of the Maurer-Cartan form of $E(n)$ and s is a local section of the bundle $E(n) \to \mathbf{R}^n$ the forms

$$\theta^A = s^*\Theta^A \tag{3}$$

give a local orthonormal coframe in $\mathbf{R}^n$ whose corresponding Levi-Civita connection forms are

$$\theta^A_B = s^*\Theta^A_B. \tag{4}$$

From now on we will drop the pull-back notation because it will be clear from the context where the forms must be considered. Let $f\colon M \to \mathbf{R}^n$ be an immersion of an m-dimensional manifold. A Darboux frame along f is a (locally defined) smooth function e on M with values in $E(n)$ of the form

$$e\colon p \to (e_A(p), f(p))$$

where $e_A(p)$ are the columns of an $\mathrm{SO}(n)$ matrix such that the vectors $e_i(p)$ span the image of the tangent space of M at p under the differential of f and determine the correct orientation. It follows that on M

$$de_A = \theta^B_A \otimes e_B, \tag{5}$$

$$\theta^\alpha = 0. \tag{6}$$

In particular (6) implies that the metric $d\sigma^2$ induced by f on M can be written as

$$d\sigma^2 = \sum_i (\theta^i)^2. \tag{7}$$

Suppose now M is an oriented Riemannian manifold with metric ds^2. Let ϕ^i be an oriented orthonormal (local) coframe on it with corresponding connection forms ϕ^i_j. On the common domain of definition of the θ^A and ϕ^i's we have

$$e^A = B^A_j \phi^j \tag{8}$$

for some smooth function B_j^A. According to (6)

$$B_j^\alpha = 0. \tag{9}$$

In particular the volume element $d\tilde{V}$ of the metric $d\sigma^2$ can be expressed as

$$d\tilde{V} = \det(B_j^i)dV \tag{10}$$

where dV is the volume element of the metric ds^2; equivalently their ratio is given by the positive function

$$u = \det(B_j^i). \tag{11}$$

The immersion f will be said to be volume decreasing if at every point $p \in M$

$$u \le 1. \tag{12}$$

Volume increasing and volume preserving are defined analogously. Exterior differentiation of (6) and (8) and use of the structure equations of $\mathbf{R}^n$ and (M, ds^2) gives:

$$dB_i^A - B_j^A \phi_i^j + B_i^B \theta_B^A = B_{ij}^A \phi^j \tag{13}$$

for some smooth functions B_{ij}^A such that $B_{ij}^A = B_{ji}^A$. The B_{ij}^A's are the coefficients of the (generalized) second fundamental tensor of the immersion $f\colon (M, ds^2) \to \mathbf{R}^n$, i.e.

$$\nabla df = B_{ij}^A \phi^i \otimes \phi^j \otimes e_A \tag{14}$$

whose trace with respect to ds^2 gives the tension field τ of f, i.e.

$$\tau = B_{ii}^A e_A. \tag{15}$$

We remark that if instead of considering $f\colon (M, ds^2) \to \mathbf{R}^n$ we consider $f\colon (M, d\sigma^2) \to \mathbf{R}^n$ the above procedure gives the second fundamental tensor and m times the mean curvature vector H.

We denote by Δ_{ds^2}, $\Delta_{d\sigma^2}$ the Laplace-Beltrami operators relative to ds^2 and $d\sigma^2$. We now claim

$$\tfrac{1}{2}\Delta_{ds^2}|f|^2 = \langle f, \tau\rangle + \|df\|^2 \tag{16}$$

and similarly

$$\tfrac{1}{2}\Delta_{d\sigma^2}|f|^2 = m\{\langle f, H\rangle + 1\}. \tag{17}$$

In the above formulas $\langle\ ,\ \rangle$ is the usual inner product in $\mathbf{R}^n$ and $|\ |$ its corresponding norm, while $\|\ \|$ is the Hilbert-Schmidt norm of df; that is

$$\|df\|^2 = \sum_{i,A}(B_i^A)^2. \tag{18}$$

The proof of (16) is a standard computation. Indeed by (6) and (8) we have

$$d|f|^2 = 2B_i^A\langle f, e_A\rangle\phi^i$$

and by (5), (6), (8), (13)

$$d(2B_i^A\langle f, e_A\rangle) - 2B_j^A\langle f, e_A\rangle\phi_i^j = 2\{\langle f, e_A\rangle B_{ij}^A + B_i^A B_j^A\}\phi^j. \tag{19}$$

By definition $\Delta_{ds^2}|f|^2$ is the trace of the coefficients appearing in the right-hand side of (19), hence by (15) and (18) we obtain (16).

In case M is compact, integration of (16) gives

$$E(f) = -\frac{1}{2}\int_M \langle f, \tau\rangle \, dV \tag{20}$$

where $E(f)$ is the energy of f. If f is an isometry (20) generalizes a formula of Minkowski on convex bodies.

3. Proof of the theorem. We just prove sufficiency. Since M is compact, integrating (16), (17) and using (10), (11) we obtain

$$\int_M \{\langle f, \tau - umH\rangle + \|df\|^2 - um\}\, dV = 0. \tag{21}$$

We now let ds^2 and $d\sigma^2$ play the role of G and H in the introduction. Our considerations will be pointwise. The matrix of ds^2 with respect to the basis ϕ^i is of course the identity (δ_{ij}), while from (7) and (8) we get

$$d\sigma^2 = B_i^k B_j^k \phi^i \phi^j$$

showing that the matrix of $d\sigma^2$ with respect to the same basis is $(B_i^k B_j^k)$. In particular from (11) its determinant is u^2. A simple computation shows that in this case $P = \frac{1}{m}\|df\|^2$. From (1) we therefore obtain

$$\|df\|^2 \geq mu^{2/m}, \tag{22}$$

and hence

$$\|df\|^2 - um \geq m(u^{2/m} - u). \tag{23}$$

On the other hand by (i) and (21) we get $\int(\|df\|^2 - um) \leq 0$. Thus, if

$$u^{2/m} - u \geq 0, \tag{24}$$

combining with (23) gives

$$u = u^{2/m}. \tag{25}$$

We deduce that equality holds in (22), hence

$$B_i^k B_j^k = \rho\delta_{ij}; \tag{26}$$

that is, the map f is conformal. Now in case $m \geq 3$ (24) follows from (j); moreover from (25) we deduce $u = 1$ which implies $\rho = 1$ in (26), i.e. f is an isometry. The remaining two cases are handled similarly.

References

1. D. Hoffman and R. Osserman, *The area of the generalized gaussian image and the stability of minimal surface in S^n and $\mathbf{R}^n$*, Math. Ann. **260** (1982), 437–452.

Department of Mathematical Sciences, University of Durham, Durham, England

International Centre for Theoretical Physics, 34100 Trieste, Italy