A NOTE CONCERNING THE 3-MANIFOLDS WHICH SPAN CERTAIN SURFACES IN THE 4-BALL

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ABSTRACT. We consider surfaces of the form $F \cup_K D$ where F is a Seifert surface and D is a slicing disk for the knot K . We show that, in general, there is no 3-manifold M which spans $F \cup_K D$ in the 4-ball such that F can be compressed to a disk in M .

This note is concerned with the following problem. Suppose K is a classical knot which is spanned by the Seifert surface F in $S^3 = \partial B^4$ and the slice disk D in B^4 . Must the surface $F \cup_K D$ bound a smoothly embedded, compact, orientable 3-manifold M (also referred to as a Seifert manifold) in which F is compressible to a disk?

This problem arises quite naturally from the proof that slice implies algebraically slice. Indeed, the proof that slice implies algebraically slice boils down to verifying that given any M as above (such an M always exists by transversality) a $1/2$ -basis of $H_1(F; \mathbf{Q})$ can be realized by curves in F which are rationally null-homologous in M . The above problem is equivalent, by Stallings' version of the Loop Theorem [S], to asking if an M can be found in which a $1/2$ -basis for $H_1(F)$ is realized by curves in F which are null-homotopic in M .

We establish in §2 that the answer to this problem is in general no. In fact, by appropriately modifying Zeeman's [Z] notion of twist spinning an arc, a collection of surfaces of the form $F \cup_K D$ are constructed in §1 for which we give necessary and sufficient conditions on $F \cup_K D$ in order for it to bound an M in which F can be compressed to a disk. These conditions are geometric in nature—yet, they are applicable.

1. The examples. This section is devoted to the construction of particular surfaces of the form $F \cup_K D$. The pertinent properties of these surfaces will be postponed until the next section.

We begin the construction of the desired surfaces with the following data:

- (1) K' is a smooth knot in S^3 ,
- (2) F' is an *incompressible* Seifert surface for K' ,
- (3) $N_i(K')$, $i = 1, 2$, denote tubular neighborhoods of K' where $N_1(K') \subset \text{int } N_2(K')$, and
- (4) n is a positive integer.

Now let $K' = A \cup B$ where A and B are compact, connected subarcs of K' satisfying $(\text{int } A) \cap (\text{int } B) = \emptyset$. We write $N_1(K')$ as the obvious decomposition $N(A) \cup N(B)$; see Figure 1. Note that $\Delta^4 = [S^3 - \text{int } N(B)] \times I$ is the 4-ball.

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FIGURE 1

FIGURE 2. The case $n = 2$ in cross section.

Under this identification, $A \times I$ becomes our desired slicing disk D and $K = \partial D$. Taking $N(D) = N(A) \times I$ we have

$$[S^3 - \text{int } N_1(K')] \times I = \text{closure } (\Delta^4 - N(D)).$$

To describe F , it suffices to show how F meets $\partial\Delta^4 - \text{int } N(K)$. In fact, henceforth we will view F as being properly embedded in $\partial\Delta^4 - \text{int } N(K)$ except when we write $F \cup_K D$. To construct F we begin by setting $F'' = F' \cap [S^3 - \text{int } N_1(K')]$. We presently construct a second surface, G , from F'' and the integer n as follows. Take a parallel copy of F'' in $S^3 - \text{int } N_1(K')$. This parallel copy is now altered in $N_2(K') - \text{int } N_1(K')$ by twisting $\partial N_1(K')$ through almost n full revolutions—so that now the boundary of the twisted copy agrees with $\partial F''$. The twisted copy is G ; see Figure 2.

We define the desired surface F by setting

$$F \cap ([S^3 - \text{int } N_1(K')] \times t) = \begin{cases} G, & t = 0, \\ \partial F'' \cap \partial N(B), & 0 < t < 1, \\ F'', & t = 1. \end{cases}$$

(Note that $\partial G = \partial F''$ so the above definition of F makes sense.)

REMARK. As noted in the introduction, the surface $F \cup_K D$ described above may also be constructed by means of Zeeman's twist spinning process. In this alternative construction, the slicing disk is (essentially) the disk in B^4 obtained by n twist spinning the arc A into its reflection. The corresponding Seifert surface is the boundary-connect-sum of F'' with its reflection. It is not difficult to show the resulting closed surface is the same as $F \cup_K D$ —the key step comes from noting what happens in ∂B^4 when the n twist spun disk is ambient isotoped to the (untwisted) spun disk.

FIGURE 3

FIGURE 4. The case $n = 2$. Note: The two dotted lines are identified under the covering projection.

2. A theorem. In addition to the notation developed in §1 we will employ the following notation here. Let X denote the n -fold cyclic cover of $S^3 - \text{int } N_1(K')$. Furthermore, let F_* denote a parallel copy of F'' in $S^3 - \text{int } N_1(K')$ which lies between F'' and the parallel copy of F'' used in the construction of G ; see Figure 3. $\tilde{F}''$, $\tilde{G}$, and $\tilde{F}_*$ are used to denote lifts of F'' , G , and F_* , respectively, to X . These lifts are chosen subject to the properties (i) $(\text{int } \tilde{F}'') \cap (\text{int } \tilde{G}) = \emptyset$, equivalently $\tilde{F}'' \cup \tilde{G}$ is an embedded closed surface in X , and (ii) $\tilde{F}''$ and $\tilde{F}_*$ enjoy the same parallel relationship in X as they do in $S^3 - \text{int } N_1(K')$; see Figure 4. Finally, let $U(\tilde{F}_*)$ denote the relative open “regular” neighborhood of $\tilde{F}_*$ in X which corresponds to the component of $X - (\tilde{F}'' \cup \tilde{G})$ containing $\tilde{F}_*$.

The purpose of this section is to establish the

THEOREM. *The following are equivalent:*

- (I) $X - U(\tilde{F}_*)$ is a handlebody.
- (II) $F \cup_K D$ bounds a smoothly embedded handlebody in Δ^4 .
- (III) $F \cup_K D$ bounds a smoothly embedded, orientable, 3-manifold, M , in Δ^4 such that F can be compressed to a disk in M .

(IV) *There is a collection of loops in the kernel of the inclusion $\pi_1(\text{int } F) \hookrightarrow \pi_1(\Delta^4 - N(D))$ which realize a $1/2$ -basis of $H_1(F)$.*

REMARKS. (A) Throughout the statement of the Theorem the word ‘smooth’ should be taken as ‘smooth with corners’.

(B) In general the failure of (IV) would immediately imply that $F \cup_K D$ cannot bound a 3-manifold in which F can be compressed to a disk. It should be noted however that we do not require in (IV) that the $1/2$ -basis be metabolic—which is a bit surprising initially.

Prior to entering into the proof of the Theorem we establish the following.

LEMMA. *Let V denote an irreducible, orientable, compact, 3-manifold with ∂V being a connected surface of genus g . Suppose a $1/2$ -basis of $H_1(\partial V)$ is realized by curves in ∂V which are null-homotopic in V . Then V is a handlebody.*

PROOF. Let $\alpha_1, \dots, \alpha_g$ denote the curves in ∂V which represent a $1/2$ -basis of $H_1(\partial V)$ and which are null-homotopic in V . We use $[\alpha_i]$ to denote the class represented by α_i in $H_1(\partial V)$. By Stallings’ version of the Loop Theorem, since $[\alpha_1] \neq 0$, there is a properly embedded disk $D_1 \subset V$ with $[\partial D_1] \neq 0$ in $H_1(\partial V)$.

Inductively, we assume that there exists (individually) properly embedded disks $D_1, \dots, D_k \subset V$ with $[\partial D_1], \dots, [\partial D_k]$ being linearly independent over $\mathbf{Z}$. Note then that there is a rank k submodule, N , of $H_1(\partial V)$ such that $H_1(\partial V)/N$ is a free module and $[\partial D_i] \in N$ for $i = 1, \dots, k$. If $k < g$, we then have that there is some $[\alpha_i] \notin N$. Again by applying Stallings’ version of the Loop Theorem there exists a properly embedded disk D_{k+1} in V with $[\partial D_{k+1}] \notin N$. That $[\partial D_1], \dots, [\partial D_{k+1}]$ should be linearly independent now follows from the fact that $H_1(\partial V)/N$ is free, hence $\mathbf{Z} \cdot [\partial D_{k+1}] \cap N = 0$. Inductively, there exist properly embedded disks $D_1, \dots, D_g \subset V$ with $[\partial D_1], \dots, [\partial D_g]$ being linearly independent over $\mathbf{Z}$. Unfortunately, we might have these disks intersecting one another.

Suppose now without loss that all intersections of the D_i are transverse. Consider $D_1 \cap D_2$. If this intersection is nonempty, let A denote an outermost arc of intersection in D_1 . Then A together with a subarc of ∂D_1 bounds a disk in D_1 whose interior misses D_2 . (We can assume that $D_1 \cap D_2$ contains no circles of intersection by standard cut and paste arguments.) We may then cut D_2 along this disk to obtain D_{21} and D_{22} , where each D_{2i} meets D_1 in at least one less arc than did D_2 . Evidently both D_{2i} are properly embedded by the outermost hypothesis on A and $[\partial D_2] = [\partial D_{21}] + [\partial D_{22}]$. This last equality implies that at least one of the collections $[\partial D_1], [\partial D_{2i}], \dots, [\partial D_g]$, $i = 1, 2$, must be linearly independent over $\mathbf{Z}$. In this fashion, we may replace D_2 with $\overline{D}_2$ such that $\overline{D}_2 \cap D_1 = \emptyset$ and $[\partial D_1], [\partial \overline{D}_2], \dots, [\partial D_g]$ are linearly independent over $\mathbf{Z}$.

We proceed in this fashion with $D_1 \cap D_3$, etc., until all the disks can be assumed to miss D_1 . We then make all the disks disjoint from $\overline{D}_2$ by letting $\overline{D}_2$ play the role that D_1 did above. Since $\overline{D}_2 \cap D_1 = \emptyset$, no new intersections of the higher indexed disks with D_1 will be created while cutting along subdisks of $\overline{D}_2$. Evidently, proceeding in this manner we obtain g properly embedded, disjoint disks in V whose boundaries represent linearly independent elements of $H_1(\partial V)$ over $\mathbf{Z}$. In particular, these boundary curves cannot setwise separate ∂V —for this would imply they are linearly dependent over $\mathbf{Z}$. It follows that upon compressing ∂V along these disks we are left with a 2-sphere. Since V is irreducible, this 2-sphere

bounds a 3-cell in V , which is the complement of the thickened 2-disks along which we have compressed ∂V . Hence V is a handlebody. $\square$

We now begin the

PROOF OF THE THEOREM. Initially, we note that (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) are trivial implications.

(I) $\Rightarrow$ (II). Let $\pi: X \rightarrow S^3 - \text{int } N_1(K')$ denote the covering projection. We now decompose $S^3 - \text{int } N_1(K')$ into possibly singular surfaces as follows: Initially we cut $S^3 - \text{int } N_1(K')$ along an open interval's worth of parallel copies of F_* , $F_* \times (-\varepsilon, \varepsilon)$ say. We then construct a Morse function

$$h: S^3 - [(\text{int } N_1(K') \cup (F_* \times (-\varepsilon, \varepsilon))] \rightarrow [-\varepsilon, \varepsilon].$$

Here we require $h^{-1}(\pm\varepsilon) = F \times \pm\varepsilon$, respectively, and the level sets of h meet what remains of $\partial N_1(K')$ in parallel copies of ∂F_* . The level sets of h together with $F_* \times (-\varepsilon, \varepsilon)$ stratify $S^3 - \text{int } N_1(K')$. This stratification of $S^3 - \text{int } N_1(K')$ lifts to a stratification of X . We now remove $\tilde{F}_* \times (-\varepsilon, \varepsilon)$ from X and identify $\tilde{F}''$ with $\tilde{F}'' \times -\varepsilon$. Also, without altering the stratification outside $\pi^{-1}(N_2(K'))$, we twist $\tilde{F}_* \times \varepsilon$ into $\tilde{G}$. Under this twisting manipulation the boundary of each stratum is identified with $\partial \tilde{F}''$ —so as to form a stratification of $X - U(\tilde{F}_*)$ which is pinched along the vertical boundary. We label these strata by C_t where $t \in [0, 1]$, $C_0 = \tilde{G}$, $C_1 = \tilde{F}''$, and $t \mapsto h \circ \pi(C_t \cap [X - \pi^{-1}(N_2(K'))])$ is a smooth, real-valued function. Note further that, if we have twisted $\tilde{F}_* \times \varepsilon$ into $\tilde{G}$ with due care, then $\pi|_{C_t}$ is a diffeomorphism onto its image for all t . We now construct an embedding of $X - U(\tilde{F}_*)$ into the closure of $\Delta^4 - N(D)$ by requiring that the image of this embedding meets $(S^3 - \text{int } N_1(K')) \times t$ in the “stratum” $\pi(C_t)$ for all $t \in [0, 1]$. This embedding immediately yields (I) $\Rightarrow$ (II).

(IV) $\Rightarrow$ (I). Let $p: [S^3 - \text{int } N_1(K')] \times I \rightarrow S^3 - \text{int } N_1(K')$ denote the canonical projection. By (IV), there exist loops $\lambda_1, \dots, \lambda_g$ in F which homotopically die in $[S^3 - \text{int } N_1(K')] \times I$ and realize a 1/2-basis of $H_1(F)$. The lifts of the loops $p(\lambda_1), \dots, p(\lambda_g)$ to $\tilde{F}'' \cup \tilde{G}$ provide inessential loops $\alpha_1, \dots, \alpha_g$ in X which represent a 1/2-basis of $H_1(\tilde{F}'' \cup \tilde{G})$. Since F' is an incompressible Seifert surface for K' , F'' is incompressible in $S^3 - \text{int } N_1(K')$ and therefore so must F_* be incompressible in $S^3 - \text{int } N_1(K')$. It follows that $\tilde{F}_*$ is incompressible in X . The standard cut and paste argument now implies that the loops $\alpha_1, \dots, \alpha_g$ are null-homotopic in $X - \tilde{F}_*$ and hence in $X - U(\tilde{F}_*)$. The lemma now applies to $X - U(\tilde{F}_*)$ to imply (I)—since Waldhausen [W] has shown that $X - U(\tilde{F}_*)$ is irreducible. $\square$

REMARK. Via the remark that concludes §1, the embedding constructed while verifying (I) $\Rightarrow$ (II) may be identified with a portion of one fiber in Zeeman's fibering theorem; see Zeeman's Lemma 6. The remaining portion of this fiber is obtained by spinning F'' into its reflection—this spinning takes place in the complementary hemisphere of S^4 .

Since many knots admit Seifert surfaces which are incompressible but have complements in S^3 other than open handlebodies (e.g. doubles of nontrivial knots) we obtain by taking $n = 1$

COROLLARY 1. *There exist infinitely many surfaces of the form $F \cup_K D$ which do not admit a Seifert manifold in which F can be compressed to a disk.*

In light of the remark that concludes §1 we have

COROLLARY 2. *The problem of whether a surface of the form $F \cup_K D$ bounds a smoothly embedded, orientable 3-manifold M such that F is compressible to a disk in M depends on the choice of D .*

For this just note that if we obtain our slicing disk via spinning, rather than twist spinning, the resulting surface bounds a “spun” handlebody. (Readers familiar with Theorem 2 of [T] might note that Corollary 2 answers the natural disk dependence question there. In general, one must allow for stabilizations.)

3. Concluding remarks. That twist spinning should have such an adverse effect on the resulting surface bounding a handlebody should not be surprising, considering Zeeman’s fibering theorem. Of course, what is essential to the purposes of this paper is the fact that once F and D are identified along K , one must consider the isotopy type of D (rel K).

A question arising from the Theorem is the following:

QUESTION. *Does there exist a nonfibered knot K spanned by an incompressible Seifert surface F such that the 2-fold branched cyclic cover of S^3 over K yields a handlebody when cut along a lift of F ?*

The answer to this question might very well be known. I do not, however, know of an example, although there appears to be no group-theoretic reason for such an example not to exist.

It is perhaps reasonable to suggest that the techniques of the Theorem can be extended in a case by case fashion to determine whether a surface of the form $F \cup_K D$ bounds a smooth 3-manifold in which F can be compressed to a disk provided D is a disk obtained by spinning. For instance, in [T] it is observed that if the initial $\tilde{K}'$ is fibered then the resulting $F \cup_K D$ bounds such a 3-manifold no matter which F is chosen. If D is not obtained by spinning the problem appears to be wide open—although the hypersurfaces constructed in Theorem 2 of [T] might be of use here.

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