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A GENERALIZATION OF THE VIETORIS-BEGLE THEOREM

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ABSTRACT. A theorem is proved which generalizes both the Vietoris-Begle theorem and the cell-like theorem for spaces of finite defomation dimension. The proof is geometric and uses a double mapping cylinder trick.

The *double mapping cylinder* $\mathrm{DM}(f)$ of a map $f\colon X' \to X$ is the space

$$X' \times [-1,1] \cup_a X \times \{-1,1\},$$

where $a(x,i) = (f(x),i)$ for $i = -1,1$. The composition $p(f \times \mathrm{id})\colon X' \times [-1,1] \to X$ (here $p\colon X \times [-1,1] \to X$ is the projection) induces $\hat{f}\colon \mathrm{DM}(f) \to X$ such that $\hat{f}^{-1}(x)$ is the suspension $\sum f^{-1}(x)$ of $f^{-1}(x)$ for all $x \in X$.

If $f\colon X' \to X$ and $A \subset X$, then $f|f^{-1}(A)\colon f^{-1}(A) \to A$ is denoted by f_A.

We are going to use a *double mapping cylinder trick* as in [$\mathbf{K_1}$, Lemma 7] or [$\mathbf{K_2}$, Lemma 2] (see also [$\mathbf{K_3}$, $\mathbf{D\text{-}S_1}$, and $\mathbf{D\text{-}S_2}$, p. 139]):

If $\hat{f}^{\#}\colon [X,E] \to [\mathrm{DM}(f),E]$ is a surjection then $f^{\#}\colon [X,E] \to [X',E]$ is an injection.

Indeed, let $g,h\colon X \to E$ be two maps such that $gf \approx hf$. Then there is a map $H\colon \mathrm{DM}(f) \to E$ such that $H(x,-1) = g(x)$ and $H(x,1) = h(x)$ for $x \in X$. Since $\hat{f}^{\#}$ is surjective, H extends over the mapping cylinder $M(\hat{F})$ of $\hat{f}$. Let $F\colon M(\hat{f}) \to E$ be an extension of H. Notice that $\hat{f}|X \times \{i\}$ is a homeomorphism for $i = -1,1$. Using this one can produce an embedding $a\colon X \times [-1,1] \to M(\hat{f})$ such that $a|X \times \{i\} = \mathrm{id}$ for $i = -1,1$. Then $F \cdot a$ is a homotopy joining g and h.

The aim of this note is to prove the following

THEOREM. *Let E be a connected CW complex and let $f\colon X' \to X$ be a closed surjective map of paracompact Hausdorff spaces. If all maps $f^{-1}(x) \to \Omega E$ are nullhomotopic for all $x \in X$ and $\hat{f}_A^{\#}\colon [A,\Omega E] \to [\mathrm{DM}(f_A),\Omega E]$ is a surjection for all closed subsets A of X, then $\hat{f}_A^{\#}\colon [A,E] \to [\mathrm{DM}(f_A),E]$ is a surjection for all closed subsets A of X.*

If $\hat{f}_A^{\#}\colon [A,E] \to [\mathrm{DM}(f_A),E]$ is a surjection for all closed subsets A of X, then the image of $f^{\#}\colon [X,E] \to [X',E]$ is the set

$$\{[g]\colon g\colon X' \to E \text{ and } g|f^{-1}(x) \approx \text{const for all } x \in S\}.$$

Before proceeding with the proof of the Theorem let us draw certain corollaries to it.

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COROLLARY 1. *Let E be a connected CW complex such that $\pi_i E = 0$ for $i > p$. If $f\colon X' \to X$ is a closed surjective map of paracompact Hausdorff spaces such that all maps $f^{-1}(x) \to \Omega^k E$ are nullhomotopic for all $x \in X$ and for $1 \le k \le p$, then $f^{\#}\colon [X, E] \to [X', E]$ is injective and its image is the set*

$$\{[g]\colon g\colon X' \to E \text{ and } g|f^{-1}(x) \approx \text{const for all } x \in X\}.$$

PROOF. $\pi_i(\Omega^p E) = \pi_{i+p} E = 0$ for $i > 0$. Therefore each component of $\Omega^p E$ is contractible, and since point inverses of $\hat{f}_A$ are connected, $\hat{f}_A^{\#}\colon [A, \Omega^p E] \to [\mathrm{DM}(f_A), \Omega^p E]$ is a surjection for each closed subset A of X. By induction we get from the Theorem that $\hat{f}_A^{\#}\colon [A, \Omega^k E] \to [\mathrm{DM}(f_A), \Omega^k E]$ is a surjection for $0 \le k \le p$ and the double mapping cylinder trick implies Corollary 1. □

REMARK. Corollary 1 strengthens and gives a simpler proof of one of the basic theorems used in the proof of Theorem 2 of [$\mathbf{K_4}$]. A statement and proof by John Walsh for compact metric spaces along with some further discussion can be found in Appendix B of [**W**].

If one interprets the reduced Čech cohomology group $\tilde{H}^q(A; G)$ as $[A, K(G, q)]$ (see [**G** or **S**]), then Corollary 1 implies the Vietoris-Begle theorem which we recall in the improved version from [**D**] (see also [**S**, p. 344]):

VIETORIS-BEGLE THEOREM. *Suppose $f\colon X' \to X$ is a closed surjective map of paracompact Hausdorff spaces such that $\tilde{H}^q(f^{-1}(x); G) = 0$ for all $x \in X$ and for $q < n$ $(n \ge 0)$. Then the sequence*

$$0 \to H^q(X; G) \xrightarrow{H^q(f)} H^q(X'; G) \xrightarrow{\gamma} \prod_{x \in X} H^q(f^{-1}(x); G)$$

is exact for $q \le n$, where γ is induced by the inclusion induced homomorphisms $\tilde{H}^q(X'; G) \to \tilde{H}^q(f^{-1}(x); G)$.

REMARK. The assertion that $\hat{f}^{\#}\colon [X, E] \to [\mathrm{DM}(f), E]$ is a surjection means that if g, h are maps from X into E and if $H\colon X' \times I \to E$ is a homotopy between gf and hf, then H is homotopic rel. $X' \times \{0, 1\}$ to a homotopy $K\colon X' \times I \to E$ which induces a homotopy $X \times I \to E$ between g and h. The statement of the Vietoris-Begle theorem can therefore be strengthened to include a corresponding assertion, as can be seen in the proof of that theorem.

COROLLARY 2. *If $f\colon X' \to X$ is a closed surjective map between paracompacta of finite defomation dimension such that $f^{-1}(x)$ has trivial shape for each $x \in X$, then f is a shape equivalence.*

PROOF. We must show that for any CW complex E, $f^{\#}\colon [X, E] \to [X', E]$ is a bijection. By Corollary 1 it is so for all E such that almost all its homotopy groups are trivial. Let $n >$ def-dim X, def-dim X' (see [$\mathbf{D\text{-}S_2}$] for a definition of the deformation dimension). By attaching k-cells for $k > n$ to E we can construct E' such that $\pi_i E' = 0$ for $i > n$. Since the n-skeletons of E and of E' coincide, the inclusion $i\colon E \to E'$ induces bijections $[X, E] \to [X, E']$ and $[X', E] \to [X', E']$. Now, $[X, E] \to [X', E]$ is a bijection because $[X, E'] \to [X', E']$ is a bijection. □

REMARK. See [$\mathbf{K_{1,3}}$, **Sh**, and $\mathbf{D\text{-}S_2}$] for other proofs of results similar to Corollary 2.

In the process of proving the Theorem we need to replace E by a homotopically equivalent space which is an ANE for paracompact spaces.

LEMMA. *Any CW complex has the homotopy type of an ANE for paracompact spaces.*

PROOF. Take a simplicial complex K such that its geometric realization $|K|$ with the metric topology is homotopy equivalent to a given CW complex (see [**G**, p. 149]). $|K|$ is considered as a subset of $l_2(K^{(0)})$ with the metric induced from $l^2(K^{(0)})$. As in [**H**, p. 107] the space $|K^{(n)}|$ is complete for each n. Therefore the subset

$$X = \bigcup_{n=0}^{\infty} |K^{(n)}| \times [n, \infty)$$

of $|K| \times [0, \infty)$ is complete in the metric induced from the obvious metric on $l_2(K^{(0)}) \times [0, \infty)$. It is clear that the inclusion $X \to |K| \times [0, \infty)$ induces isomorphisms of homotopy groups and, by the Whitehead theorem, it is a homotopy equivalence. Since every complete metric ANR is an ANE for paracompact spaces (see [**H**, pp. 84, 87]), the Lemma follows. □

PROOF OF THEOREM. Assume E is an ANE for paracompact spaces. The two statements we have to prove will have the same proof in the beginning. So let us assume that $F: Y \to X$ is a closed surjective map, where Y is paracompact Hausdorff, and suppose $g: Y \to E$ is a map such that $g|F^{-1}(x)$ is nullhomotopic for all x in X. Let $p: M(F) \to X$ be the projection.

Fix $x \in X$. Since $g|F^{-1}(x) \approx$ const, there exists an extension $g': Y \cup p^{-1}(x) \to E$ of g. Define $g'': Y \cup p^{-1}(x) \cup X \to E$ by $g''|Y \cup p^{-1}(x) = g'$ and $g''(X) = g'(x)$. g'' extends over a neighborhood U of $Y \cup p^{-1}(x) \cup X$ in $M(F)$. Choose a neighborhood V_x of x in X such that $p^{-1}(V_x) \subset U$. Having done that for all x in X, we choose a locally finite cover $\{A_s\}_{s \in S}$ consisting of closed sets which is a refinement of $\{V_x\}_{x \in X}$. Then, for each $s \in S$, we choose a map $g_s: Y \cup p^{-1}(A_s) \to E$ such that $g_s|Y = g$ and $g_s(A_s)$ is a one-point set.

Claim. Suppose $F = f$ or $F = \hat{f}$. If A is a closed subset of A_s for some s in S and $h: p^{-1}(A) \to E$ is an extension of $g|F^{-1}(A)$, then

$$h \approx g_s/p^{-1}(A) \operatorname{rel.} F^{-1}(A).$$

The Claim will be proved by showing that the map

$$c: B = F^{-1}(A) \times [-1, 1] \cup p^{-1}(A) \times \{-1, 1\} \to E$$

defined by $c(x, t) = g(x)$ for $(x, t) \in F^{-1}(A) \times [-1, 1]$, $c(X, -1) = g_s(x)$, $c(x, 1) = h(x)$ for $x \in p^{-1}(A)$, is nullhomotopic (because c will be extendible over $p^{-1}(A) \times [-1, 1]$ in this case, which will give a homotopy from g_s to h over $F^{-1}(A)$).

Assuming that the Claim holds we proceed as follows:

If $a: Y \cup p^{-1}(D) \to E$ is an extension of g (D is closed in X) and $s \in S$, then $a|p^{-1}(D \cap A_s) \approx g_s|p^{-1}(D \cap A_s) \operatorname{rel.} F^{-1}(D \cap A_s)$. Since $g_s|F^{-1}(A_s) \cup p^{-1}(D \cap A_s)$ is extendible over $p^{-1}(A_s)$, so is $a|F^{-1}(A_s) \cup p^{-1}(D \cap A_s)$. Therefore there exists an extension $a': Y \cup p^{-1}(D \cup A_s) \to E$ of a. By well-ordering S and by the transfinite induction we can construct an extension $g': p^{-1}(X) \to E$ of g.

So it remains to prove the Claim by showing that $c \approx$ const.

Case 1. $F = \hat{f}$.

Here $g_s|A =$ const impleis $h|A \approx$ const by the double mapping cylinder trick. Without loss of generality we may assume $h(A) = g_s(A)$. Now $c = c'd$, where

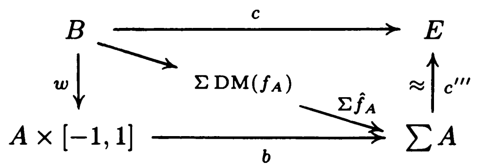
$d\colon B \to \sum \mathrm{DM}(f_A)$ is obtained by contracting each of $A \times \{-1\}$ and $A \times \{1\}$ to a point. Let $c''\colon \mathrm{DM}(f_A) \to \Omega E$ be the map induced by c'. Since

$$\hat{f}_A^{\#}\colon [A, \Omega E] \to [\mathrm{DM}(f_A), \Omega E]$$

is a surjection, there is a map $d'\colon A \to \Omega E$ such that $d'\hat{f}_A \approx c''$. This implies that there is a map $c'''\colon \sum A \to E$ such that $c'''(\sum \hat{f}_A) \approx c'$, and therefore $c \approx c'''(\sum \hat{f}_A)\, d$. However $(\sum \hat{f}_A)\, d = bw$, where $w\colon B \to A \times [-1,1]$ is induced by $\hat{f}_A \times \mathrm{id}\colon \mathrm{DM}(f_A) \times [-1,1] \to A \times [-1,1]$ (notice that B is homeomorphic to $\mathrm{DM}(\hat{f}_A)$) and $b\colon A \times [-1,1] \to \sum A$ is the quotient map. Since $b \approx$ const, we have $(\sum \hat{f}_A)\, d \approx$ const and, consequently $c \approx$ const (see the diagram). This concludes the proof of the Claim for $F = \hat{f}$. Therefore

$$\hat{f}_A^{\#}\colon [A, E] \to [\mathrm{DM}(f_A), E]$$

is a surjection.



Case 2. $F = f$.

Observe that B is homeomoprhic to $\mathrm{DM}(f_A)$. Therefore c extends over the mapping cylinder of $\hat{f}_A$. Since $c|A \times \{-1\} =$ const, any such extension is nullhomotopic. Therefore $c \approx$ const.

This completes the proof of the Theorem.

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