ON THE FROBENIUS ENDOMORPHISMS OF FERMAT AND ARTIN-SCHREIER CURVES

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ABSTRACT. The following article offers an explanation of the relationship of Jacobi and Gauss sums to Fermat and Artin-Schreier curves which is an analogue of the proof of Stickelberger's theorem.

1. Correspondences. The connection between cubic Fermat curves and cubic Jacobi sums was first observed by Gauss [G], who used it to study such sums. That one can compute the number of points on a Fermat curve over a finite field using Jacobi sums has long been known. The same is true for Artin-Schreier curves and Gauss sums and were applied by Davenport and Hasse [D-H] to compute Artin L -functions associated with such curves. This computation, in turn, yielded the Hasse-Davenport identity. Ultimately, Weil [W] interpreted these computations via a Lifshetz fixed point formula by showing that the eigenvalues of Frobenius acting on what is now known as the étale cohomology groups of Fermat curves are Jacobi sums and on Artin-Schreier curves are Gauss sums. This can be interpreted as an identity which says, in a suitable sense, that Frobenius is a Gauss or Jacobi sum (see below). One can now use this to produce annihilators of the divisor class groups of these curves which are also given by the Brumer-Stark conjecture for function fields proven in [T]. Below, we will give an elementary proof of the aforementioned identity which is analogous to Stickelberger's proof of Stickelberger's theorem. We just write down especially simple functions with appropriate divisors.

Let p be a rational prime and m an integer prime to p . Let K be a field of characteristic p . Let A_m and F_m denote the complete nonsingular curves over K with affine equations:

$$A_m: y^p - y = x^m, \quad F_m: u^m + v^m = 1.$$

Suppose $q = p^f \equiv 1 \pmod{m}$ and suppose $K \supseteq \mathbb{F}_q$. Define the homomorphisms

$$\psi: \mathbb{F}_q \rightarrow \text{Aut}(A_m), \quad \chi: \mathbb{F}_q \rightarrow \text{Aut}(A_m), \quad \chi_0, \chi_1: \mathbb{F}_q \rightarrow \text{Aut}(F_m)$$

by

$$\begin{aligned} \psi(a)(x, y) &= (x, y + T_{\mathbb{F}_q/\mathbb{F}_p}(a)), & \chi(b)(x, y) &= (b^{(q-1)/m}x, y), \\ \chi_0(b)(u, v) &= (b^{(q-1)/m}u, v), & \chi_1(b)(u, v) &= (u, b^{(q-1)/m}v) \end{aligned}$$

for $a \in \mathbb{F}_q^+$ and $b \in \mathbb{F}_q^*$.

Now recall that a correspondence between a curve X to a curve Y is a divisor Z on $X \times Y$ with no vertical or horizontal components. Among other things, Z gives

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a way of producing a divisor on Y from a divisor on X by the formula:

$$Z(D) = \pi_2(\pi_1^{-1}(D) \cdot Z),$$

where π_1 and π_2 are the first and second projections and “ $\cdot$ ” is the intersection product. In fact, this induces a homomorphism from the divisor class group of X over extensions of K to that of Y . We say two correspondences are equivalent if they induce the same such homomorphism for all such extensions. We regard a non-constant map from X to Y as the correspondence given by its graph. Finally, one can add two correspondences and if $X = Y$, one can compose two correspondences. (For more details on correspondences see [W].)

Now set

$$G_q(m) = - \sum_{b \in \mathbf{F}_q} \chi(b)\psi(b), \quad J_q(m) = - \sum_{b \in \mathbf{F}_q^* - \{1\}} \chi_0(b)\chi_1(1-b),$$

regarded as correspondences on the corresponding curve. For any curve over $\mathbf{F}_q$, ϕ_q will denote the Frobenius endomorphism (which raises the coordinates of a point to the q th power).

THEOREM A. *Under equivalence of correspondences, we have $\phi_q = G_q(m)$ on A_m and $\phi_q = J_q(m)$ on F_m .*

PROOF. We may suppose K is algebraically closed. Let ∞ denote the point at infinity on A_m . For P , a finite point on A_{q-1} over K such that $x(P) \neq 0$, let

$$g_P = (y - y(P)) - \sum_{i=0}^{f-1} (x/x(P))^{p^i}$$

in $K(A_{q-1})$. We claim that, if $x(P)^{q-1} \notin \mathbf{F}_q$,

$$(g_P) = (\phi_q - G_q(q-1))(P - \infty).$$

Indeed, first observe that g_P has pole of order p^f at ∞ and no other poles. Next, it vanishes at the $(q-1)$ points

$$\chi(b)\psi(b)P = (bx(P), y(P) + T_{\mathbf{F}_q/\mathbf{F}_p}(b)),$$

$b \in \mathbf{F}_q^*$. Finally, it vanishes at $\phi_q(P) = (x(P)^q, y(P)^q)$, which is distinct from the previous points if $x(P)^{q-1} \notin \mathbf{F}_q$. This proves the claim and establishes the theorem for A_{q-1} since it suffices to check that $(\phi_q - G_q(q-1))(P - \infty)$ is principal for all but finitely many P . (Note, however, that $x(P)^{p^{f-1}}g_P$ extends to a function which works for all P .)

Let ∞ denote any point at infinity on F_m . If Q is a finite point on F_{q-1} over K such that $u(Q) \neq 0$ and $v(Q) \neq 0$, set

$$j_Q = 1 - (u/u(Q) + v/v(Q)).$$

Then, by an argument similar to the above,

$$(j_Q) = (\phi_q - J_q(q-1))(Q - \infty)$$

if $(u(Q)^{(q-1)}, v(Q)^{(q-1)}) \notin \mathbf{F}_q$. This completes the proof for $m = q-1$. (Note, similarly to the above, $u(P)v(P)j_P$ extends to a function which works for all P .)

Now let $f: A_{q-1} \rightarrow A_m$ be the map

$$(x, y) \mapsto (x^{(q-1)/m}, y).$$

Let $N: K(A_{q-1}) \rightarrow K(A_m)$ denote the corresponding norm map on function fields. Then

$$(Ng_P) = (\phi_q - G_q(m))(f(P) - \infty),$$

which completes the proof for A_m . A similar argument completes it for F_m .

COROLLARY. *Suppose $K = \mathbb{F}_q$. Then $1 - G_q(m)$ annihilates the divisor class group of A_m and $1 - J_q(m)$ annihilates the divisor class group of F_m .*

II. Relationship to Brumer-Stark. For simplicity, we will only discuss the case of Fermat curves in this section which we consider as curves over $\mathbb{F}_q$. We will employ the notation of [T, Chapter V]. Let G_m denote the group of automorphisms of F_m generated by the images of χ_0 and χ_1 . Let l be a prime different from p . Let H denote the first l -adic étale homology group of F_m . Then it is easy to see, and well known, that

$$H \otimes_{\mathbb{Z}_l} \mathbb{Q}_l(\mu_m)$$

breaks up into one-dimensional eigenspaces for the action of G_m . In fact, if ψ is a character of G_m , then

$$(*) \quad \dim_{\mathbb{Q}_l(\mu_m)}(H^{(\psi)} \otimes \mathbb{Q}_l(\mu_m)) = \begin{cases} 0 & \text{if } \psi \circ \chi_0 = 1, \text{ or } \psi \circ \chi_0 \chi_1^{-1} = 1, \\ 1 & \text{otherwise.} \end{cases}$$

The map $(u, v) \rightarrow u$ makes F_m an Abelian cover of $\mathbb{P}^1$ with Galois group G . For a closed point x of $\mathbb{P}^1$, we let I_x denote the inertia subgroup of F_m corresponding to x . We also define the element of $\mathbb{Q}[G_m]$,

$$F_x = (1/|I_x|) \sum \sigma_v^{-1} I_v,$$

where σ_v^{-1} denotes any Frobenius element above x . In particular, $F_i = (1/m) \sum I_i$ if $i \in \{0, 1, \infty\}$. We also let $\varepsilon = (1/m^2) \sum G$. Because I_0, I_1 or I_∞ equals the kernel of χ_0, χ_1 or $\chi_0 \chi_1^{-1}$ respectively it follows from (*) that as a $\mathbb{Q}[G_m]$ module

$$(**) \quad H \approx \alpha \mathbb{Q}[G_m],$$

where $\alpha = 1 + 2\varepsilon - (F_0 + F_1 + F_\infty)$.

As in [T], we set

$$\Theta(T) = \prod (1 - F_v T^{\deg v})^{-1} \in \mathbb{Q}[G_m][[T]].$$

Using (**), Theorem A and Theorem V.2.5 of [T] we easily deduce

THEOREM B.

$$\Theta(T) = (1 - \alpha J_q(m)T)/(1 - \varepsilon T)(1 - \varepsilon qT),$$

where here we regard $J_q(m)$ as an element of $\mathbb{Q}[G_m]$.

Let $S = \{0, 1, \infty\}$, and

$$\Theta_S(T) = \Theta(T)(1 - F_0 T)(1 - F_1 T)(1 - F_\infty T).$$

Then according to Theorem V.1.2 of [T], $(q-1)\Theta(1) \in \mathbb{Z}[G_m]$ and if D is a divisor of degree zero on F_m , then there exists a function f on F_m such that

$(f) = (q-1)\Theta(1)D$ and such that the field $\mathbf{F}_q(F_m)(f^{1/(q-1)})$ is Abelian over $\mathbf{F}_q(u)$. We can see this explicitly using the results in §I.

Set $J = J_q(m)$. First of all, an easy calculation shows

$$\Theta_S(t) = (1 - \alpha J)(1 + (\varepsilon - (F_0 + F_1 + F_\infty))T + \varepsilon T^2)/(1 - \varepsilon qT)$$

and

$$\Theta_S(1) = (1 - \alpha J)\alpha = (1 - J)\alpha$$

since α is an idempotent. Moreover,

$$(1 - J)\alpha = (1 - J) + (2(q-1) - 3m)\varepsilon$$

since it is easy to check that $(1 - J)F_i = m\varepsilon$, $i \in S$. In particular, $(q-1)(1 - J)\alpha \in \mathbb{Z}[G_m]$, since m divides $q-1$.

Now let D be a divisor of degree 0 on F_m . Then we know from §I that $(1 - J)D$ is principal. Moreover, $m^2\varepsilon D$ is just the “norm” of D to $\mathbf{P}^1$, so is the divisor of a function on $\mathbf{P}^1$. It follows immediately that $(q-1)\Theta_S(1)D$ is the divisor of a function on F_m , which is an m th power times a rational function in u . Hence its m th root generates an Abelian extension of $\mathbf{F}_q(u)$ as predicted. In fact, using the results of §I, one can write this function down explicitly (although it is somewhat complicated). (Note, in general, such explicit functions have been exhibited by Hayes using Drinfeld modules [H].)

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