EMBEDDINGS OF THE MAXIMAL QUOTIENT RING OF A VON NEUMANN REGULAR RING INTO ITS COMPLETION WITH RESPECT TO A RANK FUNCTION

ANNIE VOGEL

(Communicated by Donald Passman)

ABSTRACT. Let R be a commutative von Neumann regular ring. A pseudo-rank function on R is characterized by a probability measure on the spectrum of R . We exhibit two commutative von Neumann regular rings R and R' with the same spectrum S , and two rank functions N and N' on R and R' , corresponding to the same probability measure on S , and such that the maximal right quotient ring $Q_{\max}(R)$ embeds in the N -completion $\bar{R}$ of R while $Q_{\max}(R')$ does not embed in $\bar{R}'$.

Let R be a von Neumann regular ring, and let N be a rank function on R . Let $\bar{R}$ be the N -completion of R ; $\bar{R}$ is then a regular right and left self-injective ring and one may ask whether the maximal right quotient ring $Q_{\max}(R)$ of R embeds in $\bar{R}$. Sufficient conditions are already known [1, 2].

Whenever the ring R is commutative, there exists a one-to-one correspondence between rank functions on R and probability measures on the spectrum of R [3]. So we might expect that the condition " $Q_{\max}(R)$ is isomorphic to a subring of $\bar{R}$ " only depends on the spectrum of R and on the rank function. Actually the answer is negative and we prove:

PROPOSITION 1. *There exists a commutative von Neumann regular ring R' , a rank function N on R' and a regular subring R of R' such that*

$\text{Spec } R = \text{Spec } R'$,

$Q_{\max}(R)$ is isomorphic to a subring of the N -completion of R , and

$Q_{\max}(R')$ is not isomorphic to any subring of the N -completion of R' .

Let K be a field, and let R be the set of all stationary sequences of elements of K . Then R is a regular subring of $K^{\mathbb{N}}$ and may be viewed as the set of all continuous maps of the one point compactification $\mathbb{N} \cup \infty$ of $\mathbb{N}$ with values in the discrete field K .

For any element x of $\mathbb{N} \cup \infty$, let $\mathfrak{p}_x$ be the ideal of all sequences vanishing at x . The map $x \rightsquigarrow \mathfrak{p}_x$ realizes a homeomorphism of $\mathbb{N} \cup \infty$ onto $\text{Spec } R$.

Let Q be the set of all sequences of elements of K . It is easy to see that Q is a regular self-injective ring and that R is essential in Q . Hence Q is the maximal quotient ring of R .

Let N be a rank function on R . For any element a of R , set

$$U(aR) = \{\mathfrak{p} \in \text{Spec } R / aR \not\subset \mathfrak{p}\} \quad \text{and} \quad \mu(U(aR)) = N(a).$$

Received by the editors October 20, 1986 and, in revised form, January 6, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 16A30; Secondary 16A08.

Key words and phrases. Von Neumann regular ring, N -completion, maximal quotient ring.

Then μ extends to a probability measure on $\text{Spec } R$, the support of which is $\text{Spec } R$ [3]; $\text{Spec } R$ being countable, μ is characterized by the measure of points. For all n in $\mathbf{N}$, set

$$a_n = \mu(n) = \mu(\mathfrak{p}_n) \quad \text{and} \quad a_\infty = \mu(\infty) = \mu(\mathfrak{p}_\infty).$$

Then

$$\sum_{n \in \mathbf{N}} a_n + a_\infty = 1.$$

Remark 1. Since the support of μ is $\text{Spec } R$, a_n is positive for all n .

PROPOSITION 2. *Let N be a rank function on R such that $a_\infty \neq 0$. Then Q is isomorphic to a subring of the N -completion $\bar{R}$ of R if and only if K is finite.*

PROOF. Suppose K is infinite, and suppose that Q is isomorphic to a subring of $\bar{R}$. Then there exists a rank function N' on Q which extends N , and such that R is dense in Q with respect to N' .

Let $x = (x_n)$ be a sequence of distinct elements of K . Given any $\varepsilon > 0$, there exists an element $y = (y_n)$ of R such that $N'(x - y) < \varepsilon$.

Since (y_n) is a stationary sequence, there exists an integer p such that $\forall n > p$ $x_n - y_n \neq 0$. Set

$$e_n = \begin{cases} 1 & \text{if } n > p, \\ 0 & \text{otherwise.} \end{cases}$$

The element $e = (e_n)$ is an idempotent in R and the ideal eQ is included in $(x - y)Q$. This yields $N'(e) < N'(x - y) < \varepsilon$. We have

$$N'(e) = N(e) = 1 - N(1 - e) = 1 - (a_0 + \cdots + a_{p-1}).$$

Hence $N'(e) > a_\infty$, which is a contradiction if a_∞ is a nonzero element.

Suppose K is finite and let $\mathcal{U}$ be a nonconvergent ultrafilter in $\mathbf{N}$.

For each element $x = (x_n)$ of Q , set

$$P_x = \{n \in \mathbf{N} / x_n \neq 0\},$$

$$N'(x) = \begin{cases} \mu(P_x) + a_\infty & \text{whenever } P_x \in \mathcal{U}, \\ \mu(P_x) & \text{otherwise.} \end{cases}$$

Obviously, N' is a rank function on Q . Moreover, if $x = (x_n)$ is an element of R , then

$$N(x) = \begin{cases} \mu(P_x) + a_\infty & \text{whenever } x_\infty \neq 0, \\ \mu(P_x) & \text{otherwise,} \end{cases}$$

and $x_\infty = 0$ if and only if P_x is a bounded subset of $\mathbf{N}$, which means that P_x is not an element of $\mathcal{U}$. So N' extends N .

Let $x = (x_n)$ be an element of Q . Since K is finite, the family $P_x(\alpha) = \{n \in \mathbf{N} / x_n = \alpha\}_{\alpha \in K}$ is a finite partition of $\mathbf{N}$ and there exists a unique element α in K such that $P_x(\alpha)$ is an element of $\mathcal{U}$.

Let ε be a positive number. The series $\sum_{i=1}^{\infty} a_i$ being convergent, there exists an integer p such that $\sum_{i=p}^{\infty} a_i < \varepsilon$.

Set

$$y_n = \begin{cases} x_n & \text{if } n < p, \\ \alpha & \text{otherwise.} \end{cases}$$

Then $y = (y_n)$ is an element of R and P_{x-y} , which is disjoint from $P_x(\alpha)$, and does not belong to $\mathcal{U}$. So $N'(x-y) = \mu(P_{x-y})$.

On the other hand, P_{x-y} is included in the subset $\{n \in \mathbb{N}/p < n\}$, so that $\mu(P_{x-y}) < \sum_{i=p}^{\infty} a_i$. Thus R is dense in Q with respect to N' . Then Q is isomorphic to a dense subring of $\overline{R}$.

REMARK 2. Whenever a_{∞} is nul,

$$N(\mathfrak{p}_{\infty}) = \sup\{N(x) \mid x \in \mathfrak{p}_{\infty}\} = 1.$$

So every essential ideal of R has norm one, and with N being countably additive, $Q_{\max}(R)$ is isomorphic to a subring of $\overline{R}$.

PROOF OF PROPOSITION 1. Let us consider two fields K and K' with the following properties: K is finite, K' is infinite, and K is included in K' . Let R (and R') be the ring of all stationary sequences of elements of K (of K'); R and R' are commutative von Neumann regular rings and they have the same spectrum $\mathbb{N} \cup \infty$. A probability measure on $\mathbb{N} \cup \infty$, with support precisely $\mathbb{N} \cup \infty$ and for which a_{∞} is nonzero, induces two rank functions N and N' on R and R' which are equal on R . Since a_{∞} is a nonzero element, $Q_{\max}(R)$ is isomorphic to a subring of the N -completion $\overline{R}$ of R , while $Q_{\max}(R')$ is not isomorphic to any subring of $\overline{R'}$.

REMARK 3. Whenever K is finite, the embedding of $Q_{\max}(R)$ in $\overline{R}$ is not unique: it depends on the choice of a nonconvergent ultrafilter on $\mathbb{N}$.

REFERENCES

1. K. R. Goodearl and D. Handelman, *Simple self injective rings*, Comm. Algebra **3** (1975), 797-834.
2. J. M. Goursaud and L. Jeremy, *Sur l'enveloppe injective des anneaux réguliers*, Comm. Algebra **3** (1975), 763-779.
3. A. Vogel, *Mesures de probabilité sur le spectre d'un anneau régulier et applications*, C. R. Acad. Sci. Paris **289** (1979).

INSTITUT DE MATHÉMATIQUES ET D'INFORMATIQUE, 2, RUE DE LA HOUSSINIÈRE,
44072 NANTES CEDEX, FRANCE