CHARACTERIZATIONS OF DENTING POINTS

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ABSTRACT. Let x be a PC (point of continuity) for a bounded closed convex set K of a Banach space. Then x is a denting point of K if and only if x is an extreme point (resp. strong extreme point; weak*-extreme point) of K . A new definition for denting point is also given.

Throughout this paper, K is a bounded closed convex set of a Banach space X . An element x in K is called a denting point of K if $x \notin \overline{\text{co}}(K \setminus B(x, \varepsilon))$ for all $\varepsilon > 0$ where $B(x, \varepsilon) = \{y: y \in X, \|y - x\| < \varepsilon\}$. x is a PC (point of continuity) for K if the identity mapping $(K, \text{weak}) \rightarrow (K, \text{norm})$ is continuous at x .

Recall that x is a strong extreme point of K if for any sequences $\{y_n\}$ and $\{z_n\}$ in K , $\lim_{n \rightarrow \infty} \|\frac{1}{2}(y_n + z_n) - x\| = 0$ implies that $\lim_{n \rightarrow \infty} \|y_n - x\| = 0$. x is a weak*-extreme point of K if x is an extreme point of $\tilde{K}$ where $\tilde{K}$ is the weak*-closure of K in X^{**} . A Banach space X is said to have the Kadec property (K) (resp. property (G); midpoint locally uniform rotundity) if every point on the unit sphere of X is a PC (resp. denting point; strong extreme point) for the closed unit ball of X [LL, T]. It has been proved [LLT1] that a Banach space X has property (G) if and only if X is strictly convex and X has property (K). It is easy to see that every denting point of K is a strong extreme point of K and it is known [KR] that every strong extreme point of K is a weak*-extreme point of K .

DEFINITION. An element x in K is called a very strong extreme point of K if for every sequence $\{x_n\}$ of K -valued Bochner integrable functions on $[0, 1]$, the condition

$$\lim_{n \rightarrow \infty} \left\| \int_0^1 x_n(t) dt - x \right\| = 0 \quad \text{implies} \quad \lim_{n \rightarrow \infty} \int_0^1 \|x_n(t) - x\| dt = 0.$$

It is well known that if x is a denting point of K then x is a PC for K and x is an extreme point of K . In [LLT2] (or see [R]), we show that the converse is true. In fact, we have the following results.

THEOREM. Let x be an element in a bounded closed convex set K of a Banach space. Then the following are equivalent:

- (i) x is a denting point of K .
- (ii) x is a very strong extreme point of K .
- (iii) x is a PC for K and x is an extreme point (resp. strong extreme point, weak*-extreme point) of K .

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A subset S of K is an open slice if there are a nonzero x^* in X^* and $\varepsilon > 0$ such that $S = \{x: x \in K, x^*(x) > \sup x^*(K) - \varepsilon\}$. It is well known that x is a denting point of K if and only if for any $\varepsilon > 0$ there exists a slice S of K such that $x \in S$ and diameter of S is less than ε . From the following lemma, if x is a PC for K and if x is a weak*-extreme point of K then x is a denting point of K . The authors wish to thank Professor H. P. Rosenthal for providing the lemma. We include a proof of the lemma for the sake of completeness.

LEMMA. *Let x be an element in a bounded closed convex set K of a Banach space. Then x is a weak*-extreme point of K if and only if the open slices containing x form a neighborhood base for x in the weak topology on K .*

PROOF. Suppose x is a weak*-extreme point of K . Let $V = \{y: y \in K, x_i^*(y) - x_i^*(x) < \varepsilon, i = 1, 2, \dots, n\}$ be a weak neighborhood of x in K . For $i = 1, 2$, let $M_i = \{y: y \in K, x_i^*(y) - x_i^*(x) \geq \varepsilon\}$. Suppose $x \in \overline{\text{co}}(M_1 \cup M_2)$. Then there exist $y_n \in M_1, z_n \in M_2$ and $\alpha_n \in [0, 1]$ such that $\lim_{n \rightarrow \infty} \|\alpha_n y_n + (1 - \alpha_n) z_n - x\| = 0$. Since $\tilde{K}$ is weak*-compact, we conclude that there exist y, z in $\tilde{K}$ and $0 < \alpha < 1$ such that $x = \alpha y + (1 - \alpha)z$. This contradicts the fact that x is a weak*-extreme point of K . Thus $x \notin \overline{\text{co}}(M_1 \cup M_2)$. By the Hahn-Banach theorem, there exist $x^* \in X^*$ and $\delta > 0$ such that $x^*(x) + \delta > \sup x^*(\overline{\text{co}}(M_1 \cup M_2))$. Hence if y is an element in the open slice $S = \{y: y \in K, x^*(y) > \sup x^*(K) - \delta\}$ then $x_i^*(y) - x_i^*(x) < \varepsilon, i = 1, 2$. Repeating the argument, we conclude that there is an open slice which contains x and is contained in V . The inverse implication is obvious. $\square$

PROOF OF THEOREM. (i) $\Rightarrow$ (ii). Suppose x is a denting point of K . Let $\{x_n\}$ be a sequence of K -valued Bochner integrable functions on $[0, 1]$ such that $\lim_{n \rightarrow \infty} \|\int_0^1 x_n(t) dt - x\| = 0$. Since for every $\varepsilon > 0$ there exists a simple function $y_n(t)$ such that $\|x_n(t) - y_n(t)\| < \varepsilon$ for all t , we may assume that $x_n(t)$ is a simple function. It follows that $\int_0^1 x_n(t) dt = \sum_{i=1}^{k_n} a_i(n)x_{i,n}$ where $x_{i,n} \in K, a_i(n) > 0$ and $\sum_{i=1}^{k_n} a_i(n) = 1, i = 1, 2, \dots, k_n, n = 1, 2, \dots$. Suppose that

$$\lim_{n \rightarrow \infty} \int_0^1 \|x_n(t) - x\| dt = \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} a_i(n) \|x_{i,n} - x\| \neq 0.$$

By choosing a subsequence if necessary, there exists $\varepsilon > 0$ such that $\sum_{i \in I_n} a_i(n) > \varepsilon$ where $I_n = \{i: 1 \leq i \leq k_n, \|x_{i,n} - x\| \geq \varepsilon\}, n = 1, 2, \dots$. Let $\lambda_n = \sum_{i \in I_n} a_i(n), y_n = \lambda_n^{-1} \sum_{i \in I_n} a_i(n)x_{i,n}$ and $z_n = (1 - \lambda_n)^{-1} \sum_{i \notin I_n} a_i(n)x_{i,n}$. Then $y_n, z_n \in K$ and $\sum_{i=1}^{k_n} a_i(n)x_{i,n} = \lambda_n y_n + (1 - \lambda_n)z_n$. Since x is a denting point of $K, \|y_n - x\| > \delta, n = 1, 2, \dots$, for some $\delta > 0$. This contradicts the fact that every denting point is a strong extreme point of K .

It is clear that if x is a very strong extreme point of K then x is a strong extreme point of K . On the other hand, it is proved in [LLT2] that if x is a PC for K and x is an extreme point of K then x is a strong extreme point of K . It is known that x is a weak*-extreme point if x is a strong extreme point [KR]. Finally, we use the lemma to conclude that x is a denting point of K . $\square$

REMARK 1. A Banach space X is said to have the CPCP (convex point of continuity property) if every bounded closed convex set K contains at least one PC. It is known [S] that if X has CPCP then KMP is equivalent to RNP. (Recall

a Banach space has KMP (resp. RNP) if every bounded closed convex subset of X contains at least one extreme (resp. denting) point.)

REMARK 2. It can be proved directly that if x is a PC for K and x is a strong extreme point of K then x is a very strong extreme point of K .

REMARK 3. Let F be a total subspace in X^* , and let τ be the locally convex topology on X defined by F . By replacing X^* by F (resp. the weak topology by τ), we may define the τ -denting point (resp. τ -PC point) of a bounded closed convex set K in X . It can be proved that every τ -PC, extreme point of K is a strong extreme point of K and that every τ -PC, strong extreme point of K is a τ -denting point of K .

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