NORMAL REPRESENTATIONS OF BANACH JORDAN TRIPLE SYSTEMS

T. J. BARTON, T. DANG AND G. HORN

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ABSTRACT. Some new techniques in the representation theory of Banach Jordan triple systems (i.e., JB^* -triples) are developed. These are applied to prove a Kaplansky density theorem and to show that JBW^* -triples decompose into special and exceptional parts.

In [7], L. Harris introduced a generalization of C^* -algebras which he called J^* -algebras. These are subspaces of $\mathcal{L}(H)$, the bounded operators on a Hilbert space, closed under the Jordan triple product $\{abc\} = (ab^*c + cb^*a)/2$. J^* -algebras have subsequently come to be regarded as 'concrete' versions of Banach Jordan triple systems, or JB^* -triples. These complex Banach spaces include all C^* -algebras, JB^* -algebras, and the ranges of contractive projections on these spaces. They have a ternary algebraic structure, rather than a binary one, which is determined solely from the geometry of their unit balls. Specifically, Kaup [11, 12] has shown that each bounded symmetric domain in a complex Banach space is the biholomorphic image of the unit ball of some JB^* -triple.

The study of the representations of JB^* -triples was begun by Barton and Timoney in [1] and reached a level of maturity in Friedman and Russo [4]. The present work builds on both of those efforts. In particular, we shall prove that homomorphisms of JB^* -triples have normal extensions to their biduals, that each dual JB^* -triple (a JBW^* -triple) can be split into a direct sum of 'special' and 'exceptional' ideals, and that each JBW^* -triple has a faithful normal representation into a 'concrete' JBW^* -triple with weak*-closed range. The last two results answer questions raised by B. Russo at the 1985 GPOTS conference in the affirmative, and the last result plays a role in the characterization of continuous JBW^* -triples [10].

These results are the outcome of several informal discussions held by the authors during a visit to the University of California at Irvine by the first and third authors. We wish to express our gratitude to that institution for their hospitality.

The definition and basic properties of JB^* -triples can be found in [16]. Further background information may be found in [1, 3, 4, 12, 14, 17]. We shall denote the triple product in a JB^* -triple A by $\{xyz\}$. The Peirce spaces associated with a tripotent $e \in A$ will be denoted $A_k(e)$, $k = 0, 1, 2$, i.e., $x \in A_k(e) \Rightarrow \{eex\} = kx/2$. Without fear of confusion we shall write A_1 for the open unit ball of A .

A *representation* of a JB^* -triple A in a JB^* -triple B is an (algebraic) homomorphism $\varphi: A \rightarrow B$, i.e., such that $\varphi(\{abc\}_A) = \{\varphi(a), \varphi(b), \varphi(c)\}_B$. We will show below (Lemma 1) that representations are contractive. If A and B are JBW^* -triples, then a representation is called *normal* if it is weak*-weak* continuous. The

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reference to the weak* topology is not ambiguous since a JBW^* -triple has a unique predual [1, 9]. Closed subtriples of $\mathcal{L}(H)$ are called JC^* -triples, and weak*-closed (ultraweakly or σ -weakly closed) subtriples of $\mathcal{L}(H)$ are called JW^* -triples. A is *special* if it is isometrically isomorphic (=algebraically isomorphic; see Lemma 1 and [12]) to a JC^* -triple. A is *exceptional* if every representation of A into a JC^* -triple is zero. In [4] Friedman and Russo proved a Gelfand-Naimark Theorem: Every JB^* -triple is isometrically isomorphic to a subtriple of $\mathcal{L}(H) \oplus_\infty C(S, C_6)$, where S is the Stone-Čech compactification of a discrete set and C_6 is the exceptional Cartan factor associated with the exceptional domain in $\mathbf{C}^{27}$, i.e., C_6 is the set of 3×3 hermitian matrices over the complex Cayley numbers. We shall refer to a subtriple of $\mathcal{L}(H) \oplus_\infty C(S, C_6)$ as *concrete*; thus, every JB^* -triple has a faithful representation as a concrete JB^* -triple. An equivalent formulation of this theorem allows the representation of a JB^* -triple to be a subtriple of $\mathcal{L}(H) \oplus_\infty C(S_1, C_5) \oplus_\infty C(S_2, C_6)$, where S_1 and S_2 are suitable topological spaces and C_5 is the exceptional Cartan factor associated with the exceptional domain in $\mathbf{C}^{16}$, i.e., C_5 is the set of 1×2 matrices over the complex Cayley numbers. Use of this form will allow us a more precise description of the exceptional part of a JBW^* -triple in Theorem 8 below.

LEMMA 1. *Let A and B be JB^* -triples and $\varphi: A \rightarrow B$ a homomorphism. Then*

- (a) *φ is contractive,*
- (b) *if φ is injective, then φ is an isometry.*

PROOF. Let $x \in A$ and let A_x be the norm closed subtriple of A generated by x . Then A_x is abelian (i.e., $\{ab\{cde\}\} = \{\{abc\}de\} \forall a, b, c, d, e \in A_x$), and, by continuity of the triple product in B , so is $C = \varphi(A)$. Let Λ_X be the set of homomorphisms of an abelian JB^* -triple X into $\mathbf{C}$ endowed with the relative weak*-topology, and let $\lambda \in \Lambda_C$. Then $\lambda \circ \varphi|_{A_x} \in \Lambda_{A_x}$ and so $\|\lambda \circ \varphi|_{A_x}\| \leq 1$ by [12, Lemma 1.6]. By [12, Corollary 1.11] the Gelfand map isometrically embeds C as a subtriple of $C_0(\Lambda_C)$, so $\|\varphi(x)\| = \sup_{\lambda \in \Lambda_C} |\lambda(\varphi(x))|$. Statement (a) follows.

Using the Gelfand map we may assume that A and B are commutative C^* -algebras and $\varphi: A \rightarrow B$ has dense image. By (a) φ is continuous. From the separate weak*-continuity of the triple product on A^{**} and the weak*-continuity of $\varphi^{**}: A^{**} \rightarrow B^{**}$ it follows that φ^{**} is a homomorphism with weak*-dense image. Let e be the identity of the W^* -algebra A^{**} and let $u = \varphi^{**}(e)$. Then $\varphi(A) \subseteq (B^{**})_2(u) \Rightarrow B^{**} = (B^{**})_2(u)$ and so u is a unitary element in the W^* -algebra B^{**} . Therefore $\psi: A^{**} \rightarrow B^{**}$ defined by $\psi(z) = \varphi^{**}(z)u^*$ is a $*$ -algebra homomorphism. Statement (b) now follows from a well-known result of Kaplansky (cf. [15, Corollary 1.2.6]) applied to $\psi|_A: A \rightarrow B^{**}$. $\square$

We shall make frequent use of the following result, due to Horn [9].

LEMMA 2. *The weak*-closed ideals of a JBW^* -triple are precisely its M -summands.*

PROPOSITION 3. *Let A and B be JBW^* -triples and $\varphi: A \rightarrow B$ a normal representation. Then $\varphi(A)$ is weak*-closed in B .*

PROOF. By Lemma 2 $\ker \varphi$ is an M -summand in A . Thus, there is an M -projection P on A such that $\varphi P(A) = \varphi(A)$ and φP is injective, and so is an isometric isomorphism by Lemma 1. Thus, the closed unit ball of $\varphi(A)$ is $\varphi(\overline{A_1})$ which is weak*-compact in B by Alaoglu's Theorem and the weak*-continuity of φ .

Hence, the Krein-Šmul'yan Theorem implies that $\varphi(A)$ is a weak*-closed subspace of B . $\square$

Our principal tool in this article is the following theorem which establishes the *normal extension* of a representation into a JBW^* -triple.

THEOREM 4. *Let A be a JB^* -triple, B a JBW^* -triple, and $\varphi: A \rightarrow B$ a representation. Then there is a unique normal extension of φ to a representation $\tilde{\varphi}: A^{**} \rightarrow B$.*

PROOF. Let $\iota: B_* \rightarrow B^*$ be the canonical inclusion, where B_* is the predual of B , and define $T = \iota^*$. Then T is weak*-weak* continuous and it readily follows that $T(x) = x \ \forall x \in B$. By [2] B is a subtriple of B^{**} and so $T|_B$ is a homomorphism. Since B is weak*-dense in B^{**} and $\{\cdot, \cdot, \cdot\}_{B^{**}}$ is separately weak* continuous [1], T is a homomorphism on all of B^{**} .

Define $\tilde{\varphi} = T\varphi^{**}$. Then $\tilde{\varphi}$ is weak*-weak* continuous. As in the above paragraph $\varphi^{**}: A^{**} \rightarrow B^{**}$ is a homomorphism and thus so is $\tilde{\varphi}$. For $x \in A$, $T\varphi^{**}(x) = T\varphi(x) = \varphi(x)$ so $\tilde{\varphi}$ extends φ . Any other normal extension of $\tilde{\varphi}$ must agree with $\tilde{\varphi}$ on the weak*-dense subset A and so coincides with $\tilde{\varphi}$. $\square$

Throughout the rest of this paper we shall place a tilde above a representation to indicate its normal extension.

COROLLARY 5 (KAPLANSKY DENSITY THEOREM). *Let A be a weak*-dense norm closed subtriple of a JBW^* -triple B . Then A_1 is weak*-dense in B_1 .*

PROOF. Let $\iota: A \rightarrow B$ be the inclusion map. By Proposition 3 $\tilde{\iota}(A^{**})$ is weak*-closed in B . The hypothesis then entails $\tilde{\iota}(A^{**}) = B$, and as the weak*-closure of A_1 in A^{**} is $\overline{(A^{**})_1}$ we have $\tilde{\iota}(\overline{(A_1)^{\text{weak}^*}}) = \overline{B_1}$. But

$$\tilde{\iota}(\overline{(A_1)^{\text{weak}^*}}) \subseteq \overline{\tilde{\iota}(A_1)^{\text{weak}^*}} = \overline{\iota(A_1)^{\text{weak}^*}} = \overline{A_1^{\text{weak}^*}}. \quad \square$$

For the next result it will be convenient to distinguish between a Banach space and its canonical image in its bidual. Let A be a JB^* -triple and let $j: A \rightarrow A^{**}$ be the canonical inclusion. Each weak*-closed ideal M in A^{**} is an M -summand by Lemma 2. With the M -projection $P: A^{**} \rightarrow M$ we define the *canonical representation* of A corresponding to M to be the map $x \rightarrow Pj(x)$. That this defines a representation follows from Lemma 2. Two representations $\varphi_k: A \rightarrow B_k$, $k = 1, 2$, are *equivalent* if there is an algebraic isomorphism $\psi: B_1 \rightarrow B_2$ such that $\varphi_2 = \psi\varphi_1$.

PROPOSITION 6. *Let A be a JB^* -triple, B a JBW^* -triple, and $\varphi: A \rightarrow B$ a representation with weak*-dense image. Then B is algebraically isomorphic to a weak*-closed ideal M in A^{**} and φ is equivalent to the canonical representation of A corresponding to M .*

PROOF. Since $\ker \tilde{\varphi}$ is a weak*-closed ideal in A^{**} , by Lemma 2 there is a weak*-closed ideal M in A^{**} with $A^{**} = M \oplus_{\infty} \ker \tilde{\varphi}$. Let $x \rightarrow Pj(x)$ be the canonical representation of A corresponding to M and define $\psi = \varphi|_M: M \rightarrow B$. Then ψ is a normal monomorphism onto a weak*-dense subset of B . By Proposition 3 we have in fact $\psi(M) = B$, i.e., ψ is an algebraic isomorphism of M onto B . Finally, for $x \in A$, $\varphi(x) = \tilde{\varphi}(j(x)) = \tilde{\varphi}Pj(x) = \psi Pj(x)$. $\square$

By removing the weak*-dense hypothesis in Proposition 6 and replacing B with the weak*-closure of $\varphi(A)$ in the above proof we obtain the following.

COROLLARY 7. *Let $\varphi: A \rightarrow B$ be a representation of a JB^* -triple into a JBW^* -triple. Then the weak*-closure of $\varphi(A)$ is algebraically isomorphic to a weak*-closed ideal in A^{**} . Moreover, $\overline{\varphi(A)}^{\text{weak}^*} = \tilde{\varphi}(A^{**})$.*

THEOREM 8. *Every JBW^* -triple A has a unique decomposition $A = A_{\text{sp}} \oplus_{\infty} A_{\text{ex}}$, where A_{sp} is isometrically isomorphic to a JC^* -triple and A_{ex} is isometrically isomorphic to $C(S_1, C_5) \oplus_{\infty} C(S_2, C_6)$ for some hyperstonean spaces S_1 and S_2 . A_{ex} coincides with the purely exceptional ideal of [4, Theorem 2].*

PROOF. By [9, (4.13)] A has a decomposition $A = V \oplus_{\infty} W$, where V is a type I JBW^* -triple and W is a JBW^* -triple with no nonzero abelian tripotents. V decomposes as desired by the classification theorem for type I JBW^* -triples [8]. Thus, it suffices to prove that W is isometrically isomorphic to a JC^* -triple.

By [13] the maximal tripotents (i.e., those tripotents $e \in W$ such that $W_0(e) = 0$) in a JB^* -triple coincide with the real and the complex extreme points of the closed unit ball. Since W is a dual space, we may choose a maximal tripotent $e \in W$ by the Krein-Milman Theorem. By [9] $W_2(e)$ is a JBW^* -algebra. It follows that $W_2(e)$ contains no nonzero abelian projections. Hence, $W_2(e)$ contains four strongly connected, pairwise orthogonal projections e_1, e_2, e_3, e_4 with $e_1 + e_2 + e_3 + e_4 = e$ (see [6, 5.2.15]), i.e., there are $x_{ij} \in W_1(e_i) \cap W_1(e_j)$, $i, j = 1, 2, 3, 4$, such that

$$(*) \quad \{x_{ij}, e_i, x_{ij}\} = e_j.$$

Let $\varphi: W \rightarrow C$ be a representation into an exceptional Cartan factor C (so $C = C_5$ or C_6). Suppose $\exists x \in W$ with $\varphi(x) \neq 0$. By maximality of $e \exists y \in W$ with $x = \{eey\}$ and so $0 \neq \{\varphi(e), \varphi(e), \varphi(y)\} \Rightarrow \varphi(e) \neq 0$. By (*) $\varphi(e_i)$, $i = 1, 2, 3, 4$, are either all zero or all nonzero, and since $e_1 + e_2 + e_3 + e_4 = e$ they must all be nonzero. Consequently, C has rank at least 4, a contradiction since C_5 has rank 2 and C_6 has rank 3. Thus, $\varphi = 0$ and so W is isometrically isomorphic to a JC^* -triple by [4, Corollary 4].

If $A = \tilde{A}_{\text{sp}} \oplus_{\infty} \tilde{A}_{\text{ex}}$ is another decomposition of A into special and exceptional parts, then $A_{\text{ex}} = (A_{\text{ex}} \cap \tilde{A}_{\text{ex}}) \oplus (A_{\text{ex}} \cap \tilde{A}_{\text{sp}})$ by [9, (4.2)]. But $A_{\text{ex}} \cap \tilde{A}_{\text{sp}}$ is both special and exceptional and so is zero. Thus, $A_{\text{ex}} \subseteq \tilde{A}_{\text{ex}}$, and by symmetry $A_{\text{ex}} = \tilde{A}_{\text{ex}}$ and $A_{\text{sp}} = \tilde{A}_{\text{sp}}$.

To prove the second assertion of the theorem we must show that every representation of A_{ex} into a JC^* -triple is zero. Let $\varphi: C(S_1, C_5) \oplus_{\infty} C(S_2, C_6) \rightarrow B$ be such a representation and assume $\varphi \neq 0$. Consider C_5 embedded in $C(S_1, C_5)$ as the subspace of constant functions. The tripotent $f(s) \equiv [1 \ 0]$ is maximal in $C(S_1, C_5)$ and so $\varphi(C_5) \neq 0$. Since C_5 has no proper ideal, $\varphi|_{C_5}$ is faithful, which contradicts the exceptionality of C_5 . A similar argument for C_6 yields $\varphi = 0$. $\square$

COROLLARY 9. *Every JBW^* -triple has a faithful normal representation onto a weak*-closed JBW^* -subtriple of $\mathcal{L}(H) \oplus_{\infty} C(S_1, C_5) \oplus_{\infty} C(S_2, C_6)$ for some hyperstonean spaces S_1 and S_2 . In particular, a special JBW^* -triple has a faithful normal representation as a JW^* -triple.*

PROOF. Let A be a JBW^* -triple and $A = A_{\text{sp}} \oplus_{\infty} A_{\text{ex}}$ the decomposition given by Theorem 6. A_{ex} is isometrically isomorphic to $C(S_1, C_5) \oplus_{\infty} C(S_2, C_6)$, and the normality of this representation follows from the uniqueness of preduals of JBW^* -triples (see [5, Theorem 12]). Thus, we may assume that A is special.

Let $B \subset \mathcal{L}(H)$ be a JC^* -triple and let $\varphi: B \rightarrow A$ be an algebraic isomorphism. By Proposition 6, A is algebraically isomorphic to a weak*-closed ideal in B^{**} .

Algebraic isomorphisms are surjective isometries by Lemma 1 and therefore are weak*-weak* continuous by uniqueness of preduals, so it will suffice to show that B^{**} is algebraically isomorphic to a JW^* -triple.

Let $\iota: B \rightarrow \mathcal{L}(H)$ be the inclusion map. Then $\iota^{**}: B^{**} \rightarrow \mathcal{L}(H)^{**}$ is an isometry and so ι^{**} is a weak*-weak* continuous monomorphism. By [6, 1.1.23] $\iota^{**}(A^{**})$ is weak*-closed in $\mathcal{L}(H)^{**}$. Let $\psi: \mathcal{L}(H)^{**} \rightarrow \mathcal{L}(K)$ be a faithful normal representation of $\mathcal{L}(H)^{**}$ as a von Neumann algebra, where K is some complex Hilbert space (cf. [15, Theorem 1.16.7]). Then $\psi \circ \iota^{**}$ is an algebraic isomorphism of B^{**} onto a weak*-closed subtriple of $\mathcal{L}(K)$, i.e., onto a JW^* -triple. $\square$

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DEPARTMENT OF MATHEMATICS, MEMPHIS STATE UNIVERSITY, MEMPHIS, TENNESSEE 38152 (Current address of T. J. Barton)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA-IRVINE, IRVINE, CALIFORNIA 92717 (Current address of T. Dang)

MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, D-7400 TÜBINGEN, WEST GERMANY

Current address (G. Horn): Siemens AG, Corporate Research and Technology, Otto-Hahn-Ring 6, D-8000 Munich 83, West Germany