SUMMATION METHODS AND UNIQUENESS IN VILENKIN GROUPS

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ABSTRACT. It is shown that the σ -compact U -sets for two different summation methods on Vilenkin groups are the same.

Let G be a Vilenkin group, i.e. a compact, totally disconnected, abelian, metric group, and let $\{H_n\}_{n=0}^\infty$ be a decreasing sequence of clopen subgroups of G forming a neighborhood base at the identity of G . Let Γ be the Pontryagin dual of G and let $K_n = \text{Ann}(\Gamma, H_n)$ be the annihilator of H_n in Γ . Then $\{K_n\}_{n=0}^\infty$ is an increasing sequence of finite subgroups of Γ whose union is all of Γ .

Let $A(G)$ be the algebra of absolutely convergent Fourier series, $PM(G)$ the space of pseudomeasures, and $PF(G)$ the space of pseudofunctions, on G . Then $A(G) \simeq l_1(\Gamma)$, $PM(G) \simeq l_\infty(\Gamma)$, and $PF(G) \simeq c_0(\Gamma)$, where the isomorphisms are all written as $S \mapsto \hat{S}$.

If $S \in PM(G)$, we may define the n th partial sum of the Fourier series of type 1 of S at a point x in G by

$$s_n(S, x) = s_n(S)(x) = \langle S, \xi_{xH_n} \rangle / \lambda(H_n) = \sum_{\gamma \in K_n} \hat{S}(\gamma) \gamma(x),$$

where $\langle S, f \rangle$ realizes the duality $PM(G) \simeq A(G)^*$ of Banach spaces. This summation method is investigated in [1 and 2].

In [3] Vilenkin showed how to enumerate Γ as $\{\gamma_n\}_{n=0}^\infty$ in such a way that

(*) for fixed $m \geq 0$, the sequence $\{\gamma_n\}_{n=0}^\infty$ fills cosets of K_m successively. Also $\gamma_0 = 1$.

Using this enumeration, we may construct another summation method for trigonometric series. Simply investigate the series

$$\sum_{n=0}^{\infty} \hat{S}(\gamma_n) \gamma_n(x)$$

for $S \in PM(G)$. We call partial sums of this series partial sums of type 2, and this the Fourier series of S of type 2.

We call a subset E of G a set of uniqueness or U -set of type 1 (resp. type 2) if the only Fourier series of type 1 (resp. type 2) of a pseudofunction converging to 0 everywhere except, possibly, on E is the zero series. Notice that pseudofunctions are used in this definition and not pseudomeasures. This is important since otherwise there would be no U -sets of type 1. (See [1] for a discussion.)

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The object of this paper is a demonstration that for σ -compact subsets of G , the U -sets of type 1 are exactly the U -sets of type 2. Because a partial sum of type 1 is a partial sum of type 2 (recall $(*)$ and $\gamma_0 = 1$), a U -set of type 1 is a U -set of type 2, even if not σ -compact. It is also easy to show that the empty set is a U -set of type 1, and it is known that countable unions of closed U -sets of type 1 are again U -sets of type 1. (See [1 and 2].) It is also known that closed U -sets of type 1 are exactly those closed sets which support no pseudofunction in the distributional sense [2]. Thus three different definitions of the concept of a U -set coincide for closed sets.

A key step in our program is the following proposition. For the case when $G = \Pi\mathbb{Z}/(2)$, see [4 and 5].

THEOREM 1. *Let $S \in PF(G)$ and let xH_m be a basic open set in G . Then the type 1 Fourier series of S converges to 0 on xH_m if and only if the type 2 Fourier series of S converges to 0 on xH_m .*

PROOF. Since type 1 partial sums are type 2 partial sums, one direction is trivial.

Now assume that $s_n(S, y)$ converges to 0 for every y in xH_m . Let Q be a representative set in Γ for the cosets of K_m , where K_m is the annihilator of H_m . Then for y in H_m and $n \geq m$, write

$$\begin{aligned} s_n(S, xy) &= \sum_{\gamma \in K_n} \hat{S}(\gamma) \gamma(xy) = \sum_{\phi \in Q \cap K_n} \sum_{\eta \in K_m} \hat{S}(\phi\eta)(\phi\eta)(xy) \\ &= \sum_{\phi \in Q \cap K_n} \left[\sum_{\eta \in K_m} \hat{S}(\phi\eta) \eta(x) \right] \phi(y). \end{aligned}$$

Since the Pontryagin dual of H_m is isomorphic to Γ/K_m , this last sum may be regarded as a partial sum of a type 1 Fourier series on H_m . Since $s_n(S, xy)$ converges to 0 for all $y \in H_m$, and since the empty set is a U -set of type 1 in H_m , we get

$$\sum_{\eta \in K_m} \hat{S}(\phi\eta) \eta(x) = 0 \quad \text{for } \phi \in Q.$$

If $y \in xH_m$, $yH_m = xH_m$, so the above holds with y in place of x . Rewriting gives

$$(1) \quad \sum_{\gamma \in \phi K_m} \hat{S}(\gamma) \gamma(y) = 0 \quad \text{for } y \in xH_m \text{ and } \phi \in Q.$$

Now, if $n \geq 0$ is given, then $\{\gamma_0, \dots, \gamma_n\}$ is a union of cosets of K_m except for a "tail" $\{\gamma_p, \dots, \gamma_m\}$ lying entirely in one coset of K_m , since the sequence $\{\gamma_j\}_1^\infty$ fills up cosets of K_m successively. This fact, with (1) shows

$$\begin{aligned} \left| \sum_{k=0}^n \hat{S}(\gamma_k) \gamma_k(y) \right| &= \left| \sum_{k=p}^n \hat{S}(\gamma_k) \gamma_k(y) \right| \leq (n - p + 1) \sup |\hat{S}(\gamma)| \\ &\leq \text{card}(K_m) \sup |\hat{S}(\gamma)| \rightarrow 0 \quad \text{for } y \in xH_m, \end{aligned}$$

where the supremum is over a coset of K_m which goes to infinity in Γ as n goes to infinity. This completes the proof of the proposition.

COROLLARY 2. *A σ -compact subset of G is a U -set of type 1 if and only if it is a U -set of type 2.*

PROOF. As noted above, any U -set of type 1 is a U -set of type 2 automatically. The theorem shows that a closed U -set of type 2 is a U -set of type 1. Now, if $E = \bigcup_{n=0}^{\infty} E_n$ is a σ -compact U -set of type 2, with each E_n closed in G , then each E_n is a U -set of type 2 and thus of type 1. Since countable unions of U -sets of type 1 are U -sets of type 1 (see [1 or 2]), E is a U -set of type 1.

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