GENERALIZATIONS OF A THEOREM OF NAMIOKA

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ABSTRACT. We prove a result concerning separate and joint continuity that contains a celebrated result of Namioka.

In 1974 Namioka proved a result somewhat stronger than the following:

THEOREM I [N1]. *Let B be a Čech-complete topological space, K a compact Hausdorff space and $f: B \times K \rightarrow M$ a separately continuous function into the metric space M . Then there exists a dense G_δ subset G of B such that f is (jointly) continuous at each point of $G \times K$.*

There is an obvious reformulation of this result. Suppose f, B and K are as above and $C(K)$ denotes the Banach space of continuous functions on K . Then the function f induces a function $h: B \rightarrow C(K)$ defined by $h(b) = f(b, \cdot)$ and h is continuous into $C(K)$ when $C(K)$ has the simple topology.

Our own formulation of Namioka's result, with a considerably different proof, is the following:

THEOREM II [S2]. *Suppose B is a Čech-complete space and $f: B \rightarrow X$ is a function into the Banach space X such that for any x^* in X^* that is an extreme point of the unit ball of X^* we have that $x^* \circ f$ is continuous. Then f is norm-continuous at each point of a dense G_δ subset G of B .*

We postpone until the end of the paper a proof of Theorem II.

We shall say that such a function f is $\sigma(X, \text{ext})$ continuous.

We shall show that a number of recent results concerning separate continuity are consequences of this theorem.

Two natural problems which have been investigated by various authors are the following:

(1) characterize those Banach spaces X such that for any Baire space B and any $f: B \rightarrow X$ that is weakly continuous we have that f is norm-continuous at each point of a dense G_δ set;

(2) characterize those Baire spaces for which Namioka's theorem is always valid.

We show that in all of the known cases we need only assume in (1) that f is continuous with respect to the extremal topology. We reduce a number of results concerning problems (1) and (2) to Theorem I (see Theorem 3 below).

Only Hausdorff topological spaces will be considered. For background material in topology, see [E and B].

Let T be a topological space. We shall say that T is k -analytic, written ka , if there exist a Čech-complete and Lindelöf space C and an onto continuous function

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$f: C \rightarrow T$. It is a special case of a theorem of Frolik that a space C is Čech-complete and Lindelöf if and only if there exist a Polish space M and a function $p: C \rightarrow M$ that is perfect. A function is perfect if it is continuous, onto, closed and inverse images of compact sets are compact. A topological space M is Polish if it is homeomorphic to a complete, separable metric space. We may take as the definition of a ka space the following: a topological space T is ka if there exist topological spaces C and M , M Polish, and continuous functions $f: C \rightarrow T$ and $p: C \rightarrow M$ so that f is onto and p is perfect.

A perfect map $f: T \rightarrow S$ is minimal if for any proper closed subset E of T , $f(E) \neq S$.

LEMMA 1. *Suppose that B is a Baire space, X is a Banach space and $g: B \rightarrow X$ is $\sigma(X, \text{ext})$ continuous. Suppose also that E is Čech-complete, G is a dense G_δ subset of B , $h_1: G \rightarrow E$ is continuous, $h_2: E \rightarrow X$ is $\sigma(X, \text{ext})$ continuous, and $h_2 \circ h_1(b) = g(b)$ for $b \in G$. Then g is norm-continuous at each point of a dense G_δ subset of B .*

PROOF. The hypothesis is contained in the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{g} & X \\ \uparrow i & & \uparrow h_2 \\ G & \xrightarrow{h_1} & E \end{array}$$

For $\varepsilon > 0$, let

$$\mathcal{U}_\varepsilon = \{U \mid U \text{ open in } B, \text{diam } g(U \cap G) \leq \varepsilon\}$$

and

$$U_\varepsilon = \bigcup \{U \in \mathcal{U}_\varepsilon\}.$$

Suppose $\overline{U}_\varepsilon \neq B$. The closure F of $h_1((B \setminus \overline{U}_\varepsilon) \cap G)$ is also Čech-complete. Hence, by Theorem I there exists W open, $W \cap F \neq \emptyset$ and $\text{diam } h_2(W \cap F) < \varepsilon$. Since h_1 is continuous there exists U open, $U \neq \emptyset$, $U \cap \overline{U}_\varepsilon = \emptyset$ and $h_1(U \cap G) \subseteq W$. Hence, U is in $\mathcal{U}_\varepsilon$. Contradiction.

Let $G_1 = G \cap \bigcap_{n=1}^\infty U_{1/n}$. Then G_1 is a dense G_δ subset of B and for any $\varepsilon > 0$ and any $b \in G_1$ there exists U open, $b \in U$ and $\text{diam}(g(U \cap G)) < \varepsilon$. Since balls in X are $\sigma(X, \text{ext})$ closed, $\text{diam } g(\overline{U \cap G}) \leq \varepsilon$. It is easy to see that $U \subseteq \overline{U \cap G}$. It follows that g is norm-continuous at each point of G_1 .

See [T1] for a discussion of certain compact spaces K such that $C(K)$ satisfies the hypothesis of the following result.

THEOREM 2. *Suppose that X is a Banach space and X is ka in the $\sigma(X, \text{ext})$ topology. Suppose $g: B \rightarrow X$ is a $\sigma(X, \text{ext})$ continuous function defined on the Baire space B . Then g is norm-continuous at each point of a dense G_δ subset.*

PROOF. Let C be Čech-complete and Lindelöf, let $f: C \rightarrow X$ be $\sigma(X, \text{ext})$ continuous, onto and let $p: C \rightarrow M$ be perfect onto the Polish space M . Let

$$D = \{(f(c), p(c)) \mid c \in C\} \subseteq X \times M.$$

Observe that the composed mapping

$$C \xrightarrow{f \times p} D \xrightarrow{\text{proj}} M$$

is perfect, so that $f \times p: C \rightarrow D$ and $\text{proj}: D \rightarrow M$ are both perfect. Thus D is also Čech-complete (see [E]). Let

$$H = \{(b, m) | (g(b), m) \in D\}.$$

We have that H is a closed subset of $B \times M$ and $\text{proj}_B(H) = B$ because D is closed in $X \times M$ and f is onto. A consequence of the Ryll-Nardzewski, Kuratowski theorem is that there exist a dense G_δ subset G of B and a continuous function $\lambda: G \rightarrow M$ so that $H_1 = \{(b, \lambda(b)) | b \in G\} \subseteq H$. Observe that $\text{proj}_B: H_1 \rightarrow G$ is a homeomorphism onto and we may apply Lemma 1:

$$\begin{array}{ccc} B & \xrightarrow{\quad} & X \\ \uparrow i & & \uparrow \text{proj}_X \\ G & \xrightarrow{g \times \lambda} & D \end{array}$$

The following theorem was proved by Debs [Deb], generalizing results of St. Raymond [SR], Talagrand [T2] and Deville [Dev].

THEOREM 3. *Suppose $X = C(K)$ where K is a compact Hausdorff space, $g: B \rightarrow X$ is $\sigma(X, \text{ext})$ continuous and B contains a dense subspace A that is ka. Then g is norm-continuous at each point of a dense G_δ subset of B .*

PROOF. Let Y be the smallest uniformly closed subalgebra of X that contains the constant functions and $g(A)$. Then Y is k -analytic in the $\sigma(X, K)$ topology (see [T1]). From the Gelfand-Stone theorem, Y is isometric to the $C(K_1)$ space where K_1 is also compact. It follows that Y is closed in the $\sigma(X, \text{ext})$ topology. Thus, $g(B) \subseteq Y$ and we may apply the theorem above.

We give one multivalued version of these results.

THEOREM 4. *Suppose X is a Banach space that is ka in the $\sigma(X, \text{ext})$ topology and $\varphi: B \rightarrow \mathcal{P}(X)$ is a minimal upper semicontinuous compact valued (with respect to the $\sigma(X, \text{ext})$ topology) multimap. Then $\{b \mid \varphi(b) \text{ is a point}\}$ contains a dense G_δ subset of B .*

PROOF. Let

$$G = \{(x, b) \mid x \in \varphi(b), b \in B\}$$

be the graph of φ . Let $p: G \rightarrow B$ be projection. The following are easy: p is minimal perfect, G is a Baire topological space and if H_1 is a dense G_δ subset of G then there exists a dense G_δ subset $H_2 \subseteq B$ such that $p^{-1}(H_2) \subseteq H_1$. Define $f: G \rightarrow X$ by projection. There exists a dense G_δ subset H of G such that for each $(x, b) \in H$, f is norm-continuous at (x, b) . If L is a dense G_δ subset of B such that $p^{-1}(L) \subseteq H$, then $\varphi(b)$ is a one-point set for $b \in L$.

Before giving the proof of Theorem II we need the following result which was first explicitly pointed out in [BT] with a very esoteric proof. Other proofs have been given in [K and N2]. We shall show that the result can be readily reduced to a much older result of Smulian, Eberlein and Grothendieck (see [D]); namely, the following theorem is true if X is the space $C(K)$ of continuous functions on the compact Hausdorff space K and if we identify E with K in the canonical way.

THEOREM 5. *Suppose that F is a bounded subset of the Banach space X that is relatively countably compact in the $\sigma(X, E)$ topology, where E is the set of extreme points of the unit ball of X^* . Then F is relatively compact in the weak topology.*

PROOF. Let $\{z_n\}$ be a sequence in F and suppose that F is a subset of the unit ball of X and z_0 is in X and is a $\sigma(X, E)$ cluster point of $\{z_n\}$ but is not a weak cluster point. Let $Z = [\{z_i | i \geq 0\}]$. We may find $x_j^* \in X^*$, $\|x_j^*\| \leq 1$, $1 \leq j \leq l$, $1 \leq m$ and $\varepsilon > 0$ so that

$$\bigcap_{1 \leq j \leq l} \{x \mid |x_j^*(x - z_0)| < \varepsilon\} \cap \{z_i \mid i \geq m\} = \emptyset.$$

Denote by z_j^* the restriction of x_j^* to Z . The separable representation theorem (see [Cho]) says that we may find weak* Borel probability measures μ_j on the unit ball of Z^* so that $\mu_j(E_Z) = 1$ where E_Z denotes the extreme points of the unit ball of Z^* and μ_j represents z_j for all z in Z , $z_j^*(z) = \int z^*(z) d\mu_j$. We may find weak* compact subsets T_j of E_Z such that $\mu_j(T_j) > 1 - \varepsilon/4$. Let $T = \bigcup_{1 \leq j \leq l} T_j$. Define

$$V = \bigcap_{1 \leq j \leq l} \left\{ x \mid \left| \int_T (x - z_0) d\mu_j \right| < \frac{\varepsilon}{2} \text{ for } 1 \leq j \leq l \right\}.$$

If z_i is in V for some $i \geq m$ then for each j we have that

$$\frac{\varepsilon}{2} > \left| \int_T (z_0 - z_i) d\mu_j \right| \geq |z_j^*(z_0 - z_i)| - \int_D |z_0 - z_i| d\mu_j \geq |z_j^*(z_0 - z_i)| - \frac{\varepsilon}{2}$$

where $D = E_Z \setminus T$. This is a contradiction. Thus $V \cap \{z_i \mid i \geq m\} = \emptyset$. Let $K = \{x^* \in X^* \mid \|x^*\| \leq 1 \text{ and } x^*|_Z \in T\}$. We shall show that $\{z_i|K \mid i \in \mathcal{N}\}$ is relatively weakly compact in $C(K)$. Choose any subsequence $\{y_k\}$ of $\{z_i\}$. There exists x in X that is a $\sigma(X, E)$ cluster point of $\{y_k\}$. Define for each $z^* \in T$

$$K(z^*) = \{x^* \in K \mid x^*|_Z = z^*\}.$$

The extreme points of $K(z^*)$ are extreme points of the unit ball of X^* . Since each y_k is constant on each $K(z^*)$ we have that x is also constant on each $K(z^*)$ and we have that x is a $\sigma(X, K)$ cluster point of $\{y_k\}$. From the first part of the proof we know that $\{z_i|K \mid i \in \mathcal{N}\}$ is relatively weakly compact in $C(K)$ and z_0 is a $\sigma(C(K), K)$ cluster point of $\{z_i \mid i \in \mathcal{N}\}$. Extend each measure μ_j to a measure ν_j on the unit ball of X^* such that $\mu_j(T) = \nu_j(K)$. From the theorem above some subsequence of $\{z_i \mid i \in \mathcal{N}\}$ converges weakly to z_0 in $C(K)$. We also know that for $i \geq m$

$$\sup_{1 \leq j \leq l} \left| \int_K (z_i - z_0) d\nu_j \right| = \sup_{1 \leq j \leq l} \left| \int_T (z_i - z_0) d\mu_j \right| > \frac{\varepsilon}{2}$$

which is a contradiction. This proves that any countable subset of F has a weak cluster point and the theorem above shows that F is relatively weakly compact.

Analogous to [Cho] (see also [Chr]), we define a strategy to be a function s from the nonempty open subsets $\mathcal{U}$ of T to $\mathcal{U}$ with the following property:

(*) $s(U) \subseteq U$ and if $\{U_n\}$ is a sequence of open sets such that $U_{n+1} \subseteq s(U_n)$ and $t_n \in U_n$, then the sequence $\{t_n\}$ has a cluster point in $\bigcap_{n=1}^{\infty} U_n$.

It is easy to see that Čech-complete spaces have property (*).

We reformulate Theorem II in the following way:

THEOREM 6. *Let T be a topological space satisfying $(*)$ and $f: T \rightarrow X$ be a $\sigma(X, E)$ continuous function. Then, there exists a dense G_δ subset G of T such that f is continuous with respect to the norm topology at each point of G .*

PROOF. (Exactly as in [S1].) As usual, define $T_n = \bigcup \{U \mid U \text{ open, } \text{diam} f(U) \leq 2^{-n}\}$. Each T_n is open and if each T_n is dense then $G = \bigcap_{n=1}^{\infty} T_n$ is the desired set. Also, for each m ,

$$\{t \mid \|f(t)\| \leq m\} = \bigcap \{t \mid e^*(f(t)) \leq m, e^* \in E\}$$

is a closed set. Since T is a Baire space we may assume there exist a nonempty open set $U_{0,0}$ and constants $C > 0$, $\delta > 0$ such that

$$\|f(t)\| \leq C \quad \text{for all } t \in U_{0,0}$$

and $\text{diam } f(U) > \delta$ for all $\emptyset \neq U$ open, $U \subseteq U_{0,0}$.

The proof proceeds as in the usual "Cantor-space" construction. Since

$$\text{diam } f(U_{0,0}) > \delta$$

and the set of extreme points are norming, we may choose an extreme point $e_{0,0}^*$ and $r_{0,0}$ real such that

$$\begin{aligned} s(U_{0,0}) \cap \{t \mid e_{0,0}^*(f(t)) < r_{0,0}\} &= U_{1,0}, \\ s(U_{0,0}) \cap \{t \mid e_{0,0}^*(f(t)) > r_{0,0} + \delta\} &= U_{1,1} \end{aligned}$$

are both nonempty. Repeat this argument for $s(U_{1,0})$ and $s(U_{1,1})$. We obtain a sequence of extreme points

$$\{e_{n,i}^* \mid 0 \leq n < \infty, 0 \leq i < 2^n - 1\},$$

a sequence of real numbers

$$\{r_{n,i} \mid 0 \leq n < \infty, 0 \leq i < 2^n - 1\},$$

and a sequence of nonempty open subsets

$$\{U_{n,i} \mid 0 \leq n < \infty, 0 \leq i < 2^n - 1\}$$

of T such that

$$U_{n+1,2i} \cap U_{n+1,2i+1} \subseteq s(U_{n,i})$$

and

$$\sup_{t \in U_{n+1,2i}} e_{n,i}^*(f(t)) \leq r_{n,i}, \quad \inf_{t \in U_{n+1,2i+1}} e_{n,i}^*(f(t)) \geq r_{n,i} + \delta.$$

Define

$$S = \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{2^n-1} s(U_{n,i}).$$

It is easy to see (exactly as in the classical Bolzano-Weierstrass theorem) that each sequence in S has a cluster point in S ; $f(S)$ has the same property with respect to the $\sigma(X, E)$ topology. By the result above, $f(S)$ is weakly compact. Let Y be the smallest, norm-closed linear subspace of X^* containing the vectors $\{e_{n,i}^*\}$; Y is a separable Banach space. The induced function from S to Y^* is $\sigma(Y^*, Y)$

continuous. Since $f(S)$ is bounded and weakly compact it is also $\sigma(Y^*, Y)$ compact. The $\sigma(Y^*, Y)$ topology on $f(S)$ is metrizable, thus $f(S)$ is weakly separable and by the Hahn-Banach theorem it is also norm-separable. This is a contradiction because S contains an uncountable set S_0 (indexed by the “branches” of $\{(n, i) \mid n = 0, \dots, 0 \leq i < 2^n\}$) such that for two distinct points s and t in S_0 ,

$$\|f(s) - f(t)\|_{Y^*} \geq \sup_{n,i} \{ |e_{n,i}^*(f(s) - f(t))| \} \geq \delta.$$

We mention that the above proof also works for multivalued maps: suppose that T and X are as above and that $F: T \rightarrow X$ is an upper semicontinuous and compact valued map (understood that X has the $\sigma(X, E)$ topology). Then there exists a function $f: T \rightarrow X$ and a dense G_δ subset G of T such that $f(t) \in F(t)$ for all t and f is continuous at each point of G . We may assume that F is minimal (Zorn’s Lemma); the proof then proceeds exactly as above (by $F(A)$ we mean $\bigcup \{F(t) \mid t \in A\}$). Note that if F is minimal, U, V are open subsets of T and X , respectively, and $F(U) \cap V \neq \emptyset$, then there exists an open W with $U \supseteq W \neq \emptyset$, such that $F(W) \subseteq V$; one need only bear this in mind in the choice of $U_{n,i}$.

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