A RECONSIDERATION OF THE GENERAL PARABOLA THEOREM IN CONTINUED FRACTIONS

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ABSTRACT. A direct explicit proof, using one technical lemma and the difference equations, is given of the general parabola theorem.

1. Introduction. For background and further references to the general parabola theorem see [1, p. 105] and Wall's book [2]. Let

$$(1.1) \quad |a_n| - \operatorname{Re}(a_n e^{-2i\alpha}) \leq \frac{1}{2} \cos^2 \alpha$$

with $-\pi/2 < \alpha < \pi/2$ and $a_n \neq 0$ for $n \geq 1$. Then the general parabola theorem states that

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{a_n}{1}$$

converges to a finite value if and only if the Stern-Stolz series

$$(1.3) \quad \sum_{n=1}^{\infty} \prod_{k=1}^n |a_k|^{(-1)^{n+k-1}}$$

diverges. For (1.3) see [1, pp. 35, 79]. We will make use of the fact that

$$(1.4) \quad \operatorname{Re} \frac{a_n e^{-i\alpha}}{1+z} \geq -\frac{1}{2} \cos \alpha$$

for $\operatorname{Re}(ze^{-i\alpha}) \geq -\frac{1}{2} \cos \alpha$ (see [1, p. 75]). A direct proof of (1.4) is obtained by observing that

$$(1.5) \quad \inf_{t \text{ real}} \frac{x+ty}{1+t^2} = \frac{1}{2}x - \frac{1}{2}\sqrt{x^2+y^2}$$

where x and y are arbitrarily fixed real numbers. Inequality (1.5) is an exercise in calculus. Further we will use extensively the difference equations

$$(1.6) \quad A_n = A_{n-1} + a_n A_{n-2}, \quad B_n = B_{n-1} + a_n B_{n-2}$$

for $n \geq 1$ with $A_0 = B_{-1} = 0$ and $A_{-1} = B_0 = 1$. Here A_n and B_n are the numerators and denominators respectively of the n th approximant of (1.2) for $n \geq 1$ (see [1, p. 20]). We introduce now the set $D = \{z | \operatorname{Re}(ze^{-i\alpha}) \geq -\frac{1}{2} \cos \alpha\}$. The method for proving the general parabola theorem is based on the application of the

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difference equations (1.6) except for the following starting estimate:

$$\begin{aligned}
 S_n &= \sup_{z \in D} \left| \frac{A_n + zA_{n-1}}{B_n + zB_{n-1}} - \frac{A_n}{B_n} \right| = \sup_{z \in D} \frac{|z| \prod_{k=1}^n |a_k|}{|B_n| |B_n + zB_{n-1}|} \\
 &= \sup_{t \text{ real}} \frac{\prod_{k=1}^n |a_k|}{|B_n| \sqrt{|B_{n-1}|^2 + \frac{|B_n|^2 - \operatorname{Re}(\bar{B}_n B_{n-1} e^{i\alpha}) \cos \alpha - 2t \operatorname{Im}(\bar{B}_n B_{n-1} e^{i\alpha})}{((\cos^2 \alpha)/4) + t^2}}} \\
 (1.7) \quad &= \frac{(\cos \alpha) \prod_{k=1}^n |a_k|}{|B_n| ||B_n| - |B_n e^{-i\alpha} - B_{n-1} \cos \alpha||} \\
 &= \frac{\prod_{k=1}^n |a_k|}{|T_n|} \left(1 + \sqrt{1 - \frac{T_n \cos \alpha}{|B_n|^2}} \right)
 \end{aligned}$$

where $T_n = 2 \operatorname{Re}(\bar{B}_n B_{n-1} e^{i\alpha}) - |B_{n-1}|^2 \cos \alpha$ and $n \geq 1$. We have not yet justified the calculations in (1.7). This will be done in the following way. For $n \geq 2$ consider

$$\begin{aligned}
 T_n - |a_n| T_{n-1} &= |B_{n-1}|^2 \cos \alpha + |B_{n-2}|^2 |a_n| \cos \alpha \\
 &\quad + (a_n e^{-2i\alpha} - |a_n|) \bar{B}_{n-1} B_{n-2} e^{i\alpha} + (\bar{a}_n e^{2i\alpha} - |a_n|) B_{n-1} \bar{B}_{n-2} e^{-i\alpha} \\
 &\geq |B_{n-1}|^2 \cos \alpha + |B_{n-2}|^2 |a_n| \cos \alpha - 2|a_n e^{-2i\alpha} - |a_n|| |B_{n-1}| |B_{n-2}| \\
 &\geq |B_{n-1}|^2 \cos \alpha + |B_{n-2}|^2 |a_n| \cos \alpha - 2|B_{n-1}| |B_{n-2}| \cos \alpha \sqrt{|a_n|} \\
 &= (|B_{n-1}| - |B_{n-2}| \sqrt{|a_n|})^2 \cos \alpha \geq 0.
 \end{aligned}$$

Here we have used the second difference equation in (1.6) and the fact that $|a_n e^{-2i\alpha} - |a_n|| \leq (\cos \alpha) \sqrt{|a_n|}$ which follows from (1.1). Now, since $T_1 \cos \alpha > 0$, we obtain that $T_n > 0$ for $n \geq 1$ which also gives us automatically that $B_n \neq 0$ for $n \geq 1$. Since further

$$(1.8) \quad |B_n|^2 - |B_n e^{-i\alpha} - B_{n-1} \cos \alpha|^2 = T_n \cos \alpha$$

for $n \geq 1$ we conclude that $|B_n| - |B_n e^{-i\alpha} - B_{n-1} \cos \alpha| > 0$ for $n \geq 1$. The calculations in (1.7) are now easily justified. There are four steps in (1.7) which need justification. We do this justification backwards. The fourth step follows by direct calculation using (1.8). The third step follows by the use of the elementary estimate (1.5). In (1.5) we put

$$\begin{aligned}
 x &= \frac{4}{\cos^2 \alpha} (|B_n|^2 - \operatorname{Re}(\bar{B}_n B_{n-1} e^{i\alpha}) \cos \alpha), \\
 y &= -\frac{4}{\cos \alpha} \operatorname{Im}(\bar{B}_n B_{n-1} e^{i\alpha})
 \end{aligned}$$

and the third step is justified at once. The second step follows by putting $ze^{-i\alpha} = u + it$ and observing that the supremum is obtained on the line $u = -\frac{1}{2} \cos \alpha$. The first step is trivial (see [1, p. 20]). Thus all calculations in (1.7) are valid. We can now assert that

$$(1.9) \quad \left| \sum_{k=1}^{n+p} \frac{a_k}{1} - \sum_{k=1}^n \frac{a_k}{1} \right| \leq S_n$$

for $n \geq 1$ and $p \geq 0$. This is obtained by putting $z = K_{k=n+1}^{n+p} a_k / 1$ for $p \geq 1$ in (1.7) since here $\operatorname{Re}(ze^{-i\alpha}) \geq -\frac{1}{2} \cos \alpha$ because of (1.4) (see also [1, p. 20]). If now $\lim_{n \rightarrow \infty} S_n = 0$ we are dealing with a Cauchy sequence in (1.9) and we conclude

that (1.2) converges to a finite number. The supremum determination in (1.7) is closely related to the diameter determination in [1].

There are two reasons for using the more refined estimate in (1.7). The first reason is that our approach is not geometric. The other reason is that the expression $B_n e^{-i\alpha} - B_{n-1} \cos \alpha$ in (1.7) plays a crucial role later in the proof and it is interesting to note that it appears naturally at the start. On the other hand, whether we use diameter calculation or supremum calculation, we obtain at the start an expression to study in the light of the difference equations (1.6) and the rest of the proof of the parabola theorem is therefore technical routine work based on these difference equations only. This is the main point so far. It should, however, be noted that the diameter in [1] formulated in the present notation is

$$(1.10) \quad \frac{2 \prod_{k=1}^n |a_k|}{T_n}.$$

For (1.10) see [1, p. 102]. With the above background in mind, we will proceed with the actual proof of the parabola theorem.

2. The general parabola theorem. We will first prove the following lemma.

LEMMA. For $n \geq 1$ we have

$$(2.1) \quad \frac{T_n}{|a_n|} \geq \frac{2}{\cos \alpha} |B_{n-1}| (|B_{n-1}| - |B_{n-1} e^{-i\alpha} - B_{n-2} \cos \alpha|)$$

supposing condition (1.1).

PROOF. For $n = 1$ the inequality in (2.1) is trivial. Therefore let $n \geq 2$. Introducing $a_n e^{-2i\alpha} = u + it \neq 0$ we find from (1.1) that $t^2 \leq \frac{1}{4} \cos^4 \alpha + u \cos^2 \alpha$. Consider the expression

$$(2.2) \quad \begin{aligned} \frac{T_n}{|a_n|} &= \frac{|B_{n-1}|^2 \cos \alpha + a_n \bar{B}_{n-1} B_{n-2} e^{-i\alpha} + \bar{a}_n B_{n-1} \bar{B}_{n-2} e^{i\alpha}}{|a_n|} \\ &= \frac{|B_{n-1}|^2 \cos \alpha + 2u \operatorname{Re}(\bar{B}_{n-1} B_{n-2} e^{i\alpha}) - 2t \operatorname{Im}(\bar{B}_{n-1} B_{n-2} e^{i\alpha})}{\sqrt{u^2 + t^2}} \\ &= \frac{x_1 + ux_2 + tx_3}{\sqrt{u^2 + t^2}} \end{aligned}$$

with $x_1 = |B_{n-1}|^2 \cos \alpha$, $x_2 = 2 \operatorname{Re}(\bar{B}_{n-1} B_{n-2} e^{i\alpha})$, $x_3 = -2 \operatorname{Im}(\bar{B}_{n-1} B_{n-2} e^{i\alpha})$. We will first consider (2.2) for $u^2 + t^2 \neq 0$ in general. Putting the partial derivatives equal to zero we get

$$(2.3) \quad x_2(u^2 + t^2) = u(x_1 + ux_2 + tx_3), \quad x_3(u^2 + t^2) = t(x_1 + ux_2 + tx_3).$$

Combining the equations in (2.3) we find $t = x_3 u / x_2$ since $x_2 \neq 0$. Substituting this value for t in the first equation in (2.3) we get $x_1 = 0$ which is clearly wrong since obviously $x_1 \neq 0$. Thus (2.2) has no local minimum for $u^2 + t^2 \neq 0$. Consider further (2.2) for $t^2 \leq \frac{1}{4} \cos^4 \alpha + u \cos^2 \alpha$. If $u \rightarrow \infty$ we find that (2.2) tends to x_2 . Therefore the infimum of (2.2) is obtained for $t^2 = \frac{1}{4} \cos^4 \alpha + u \cos^2 \alpha$. For

$t^2 = \frac{1}{4} \cos^4 \alpha + u \cos^2 \alpha$ we find that

$$\begin{aligned}
 \frac{x_1 + ux_2 + tx_3}{\sqrt{u^2 + t^2}} &= x_2 + \frac{x_1 \cos^2 \alpha - \frac{1}{2}x_2 \cos^4 \alpha + tx_3 \cos^2 \alpha}{\frac{1}{4} \cos^4 \alpha + t^2} \\
 (2.4) \qquad &\geq \frac{2x_1}{\cos^2 \alpha} - \sqrt{\left(\frac{2x_1}{\cos^2 \alpha} - x_2\right)^2 + x_3^2} \\
 &= \frac{2}{\cos \alpha} |B_{n-1}| (|B_{n-1}| - |B_{n-1}e^{-i\alpha} - B_{n-2} \cos \alpha|).
 \end{aligned}$$

The inequality in (2.4) is established by the use of the elementary estimate (1.5). In (1.5) we put $x = 4x_1/\cos^2 \alpha - 2x_2$, $y = 2x_3$ and the inequality follows. The last equality in (2.4) is established by inserting the values for x_1, x_2, x_3 and the proof is complete.

The lemma will be used in the following theorem.

THEOREM 1 (EVEN AND ODD PARTS). *Even and odd parts of the continued fraction (1.2) converge to finite values, supposing condition (1.1).*

PROOF. We discriminate between two cases. The first case is when $\lim_{n \rightarrow \infty} S_n = 0$ in (1.7). This case is simple in the light of (1.9). The second case is when $\lim_{n \rightarrow \infty} S_n = 0$ does not hold. This case is also simple by using the standard transformation of a continued fraction into a series. We consider now the first case. From (1.9) we see that the continued fraction (1.2) converges to a finite value. Thus the even and odd parts converge to finite values which in this case are equal. Next, we will consider the second case. First, using

$$|B_{n-1}e^{-i\alpha} - B_{n-2} \cos \alpha| = \sqrt{|B_{n-1}|^2 - T_{n-1} \cos \alpha},$$

derived from (1.8), together with (2.1) in the lemma we find that

$$(2.5) \qquad \frac{T_n}{|a_n|} \geq \frac{2T_{n-1}}{1 + \sqrt{1 - T_{n-1} \cos \alpha / |B_{n-1}|^2}}$$

for $n \geq 2$. Inequality (2.5) leads immediately to

$$(2.6) \qquad \frac{|a_n|T_{n-1}}{T_n} \leq 1 - \frac{T_{n-1} \cos \alpha}{4|B_{n-1}|^2}.$$

Multiplying left and right sides respectively in (2.6) we see that

$$(2.7) \qquad \frac{S_n \cos^2 \alpha}{2|a_1|} \leq \frac{(\cos \alpha) \prod_{k=1}^n |a_k|}{|a_1|T_n} \leq \prod_{k=2}^n \left(1 - \frac{T_{k-1} \cos \alpha}{4|B_{k-1}|^2}\right)$$

for $n \geq 2$. The left inequality in (2.7) follows from (1.7). If the series

$$(2.8) \qquad \sum_{n=1}^{\infty} \frac{T_n}{|B_n|^2}$$

diverges we clearly have from (2.7) that $\lim_{n \rightarrow \infty} S_n = 0$ which contradicts the second case. Thus (2.8) converges in this case. We observe further that for $n \geq 1$

we have

$$\begin{aligned}
 (2.9) \quad \left| \frac{T_n}{|B_n|^2} - \cos \alpha \right| &= \left| \left(\frac{B_{n-1}}{B_n} - 1 \right) e^{i\alpha} + \left(\frac{\bar{B}_{n-1}}{\bar{B}_n} - 1 \right) e^{-i\alpha} - \left(\left| \frac{B_{n-1}}{B_n} \right|^2 - 1 \right) \cos \alpha \right| \\
 &\leq 2 \left| \frac{B_{n-1}}{B_n} - 1 \right| + \left| \frac{B_{n-1}}{B_n} - 1 \right| \left(1 + \left| \frac{B_{n-1}}{B_n} \right| \right) \\
 &\leq \left| \frac{B_{n-1}}{B_n} - 1 \right| \left(3 + \left| \frac{B_{n-1}}{B_n} \right| \right).
 \end{aligned}$$

Because of convergence of (2.8) we have $T_n/|B_n|^2 \leq \frac{1}{2} \cos \alpha$ for sufficiently large n . Thus for sufficiently large n we have from (2.9) that

$$(2.10) \quad \frac{1}{|B_{n-1}/B_n - 1|} \leq \frac{10}{\cos \alpha}$$

in case $|B_{n-1}/B_n| \leq 2$. If $|B_{n-1}/B_n| \geq 2$ the inequality (2.10) is trivial. Thus (2.10) is valid for sufficiently large n . We will finally prove that the series

$$(2.11) \quad \sum_{n=2}^{\infty} \left| \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} \right|$$

converges. Clearly for $n \geq 2$ we have

$$\begin{aligned}
 (2.12) \quad \left| \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} \right| &= \frac{\prod_{k=1}^{n-1} |a_k|}{|B_n| |B_{n-2}|} \\
 &= \frac{\prod_{k=1}^n |a_k|}{T_n} \cdot \frac{T_n}{|B_n|^2} \cdot \frac{1}{|B_{n-1}/B_n - 1|}.
 \end{aligned}$$

The first equality in (2.12) follows since $A_n B_{n-2} - B_n A_{n-2} = A_{n-1} B_{n-2} - B_{n-1} A_{n-2}$ for $n \geq 1$ (see also [1, p. 20]). The second equality in (2.12) is trivial. Since obviously the first factor in the last expression in (2.12) is bounded because of the last inequality in (2.7) and the same is true for the last factor for sufficiently large n in the light of (2.10), we obviously have that (2.11) converges. Since (2.11) converges we can conclude that even and odd parts of (1.2) converge to finite values. We need only observe that

$$\frac{A_{2n}}{B_{2n}} = \sum_{k=1}^n \left(\frac{A_{2k}}{B_{2k}} - \frac{A_{2k-2}}{B_{2k-2}} \right)$$

and

$$\frac{A_{2n+1}}{B_{2n+1}} = \frac{A_1}{B_1} + \sum_{k=1}^n \left(\frac{A_{2k+1}}{B_{2k+1}} - \frac{A_{2k-1}}{B_{2k-1}} \right) \quad \text{for } n \geq 1.$$

The theorem is therefore proved.

We will now use Theorem 1 in proving the next theorem.

THEOREM 2 (DIVERGENCE). *If the continued fraction (1.2) diverges we have convergence of the Stern-Stolz series (1.3), supposing condition (1.1).*

PROOF. The result is established by substituting

$$(2.13) \quad a_n = \left(1 - \frac{B_{n-1}}{B_n} \right) \frac{B_n}{B_{n-2}}$$

for $n \geq 2$ in the Stern-Stolz series (1.3) and checking if this series converges. The equality (2.13) follows from the second difference equation in (1.6). But first we notice that from Theorem 1 we have

$$(2.14) \quad \lim_{n \rightarrow \infty} \frac{A_{2n}}{B_{2n}} \neq \lim_{n \rightarrow \infty} \frac{A_{2n+1}}{B_{2n+1}}$$

supposing divergence of the continued fraction (1.2). From (2.14) we conclude that

$$(2.15) \quad \begin{aligned} 0 < \beta &\leq \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \frac{\prod_{k=1}^n |a_k|}{|B_n| |B_{n-1}|} \\ &= \frac{\prod_{k=1}^n |a_k|}{T_n} \cdot \frac{T_n}{|B_n|^2} \cdot \frac{1}{|B_{n-1}/B_n|} \end{aligned}$$

for sufficiently large n where $\beta > 0$ is a suitable constant. From (2.15) we conclude that

$$(2.16) \quad \left| \frac{B_{n-1}}{B_n} \right| \leq \frac{\prod_{k=1}^n |a_k|}{\beta T_n} \cdot \frac{T_n}{|B_n|^2} \leq \frac{|a_1|}{\beta \cos \alpha} \cdot \frac{T_n}{|B_n|^2}$$

for sufficiently large n . The last inequality in (2.16) follows from the last inequality in (2.7). We observe also that the series

$$(2.17) \quad \sum_{n=1}^{\infty} \left| \frac{B_{n-1}}{B_n} \right|$$

converges. The reason is that the series (2.8) converges. The series (2.8) converges because divergence of (2.8) implies $\lim_{n \rightarrow \infty} S_n = 0$ in light of (2.7). Thus the continued fraction (1.2) converges which is obviously wrong. Further we have by induction from (2.13) that

$$(2.18) \quad \prod_{k=1}^n |a_k|^{(-1)^{n+k-1}} = |a_1|^{(-1)^n} \cdot \left| \frac{B_{n-1}}{B_n} \right| \cdot \prod_{k=2}^n \left| 1 - \frac{B_{k-1}}{B_k} \right|^{(-1)^{n+k-1}}$$

for $n \geq 2$. The last product in (2.18) is clearly bounded in light of the convergence of (2.17). Again because of convergence of (2.17) we see that the Stern-Stolz series (1.3) converges and we are through. Finally we have the following theorem.

THEOREM 3 (THE GENERAL PARABOLA THEOREM). *The continued fraction (1.2) with condition (1.1) converges to a finite number if and only if the Stern-Stolz series (1.3) diverges.*

PROOF. If (1.2) converges, then (1.3) diverges. For this fact see [1, p. 79]. If (1.3) diverges, then (1.2) converges by Theorem 2. This concludes the proof.

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