FACTORIZATION IN OPERATOR ALGEBRAS

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ABSTRACT. Let $M \subseteq B(H)$ be a σ -finite von Neumann algebra and let $\mathcal{L}$ be a lattice of projections associated with a unitary representation $\{W_t\}$ of a compact group with an ordered dual. ($\mathcal{L}$ is not necessarily contained in M .) Assume that M is invariant under $\text{ad } W_t$ for every t .

Then, whenever T is an invertible operator in M that can be factored as $T = UA$ where U is a unitary operator in $B(H)$ and A lies in $\text{alg } \mathcal{L} \cap (\text{alg } \mathcal{L})^{-1}$, then U and A can be chosen in M .

As corollaries we derive results about factorization with respect to CSL subalgebras and analytic subalgebras of von Neumann algebras.

1. Introduction. Let M be a σ -finite von Neumann algebra and $\mathcal{T} \subseteq M$ be a σ -weakly closed (nonselfadjoint) subalgebra of M . The *factorization problem* with respect to $(M, \mathcal{T})$ is the following problem.

Given an invertible operator T in M , can we write $T = UA$ where U is a unitary operator in M and A lies in $\mathcal{T} \cap \mathcal{T}^{-1}$?

This problem is closely related to the problem of similarity among algebras of operators (see [5 and 2]). Consider the following example. Let $\mathbf{T}$ denote the unit circle, H be $L^2(\mathbf{T})$, $\{e_n\}_{n=-\infty}^{\infty}$ be the usual orthonormal basis $e_n = z^n$ and p_j be the projection onto the closed subspace spanned by $\{e_k: k \geq j\}$, $-\infty < j < \infty$. Write $\mathcal{L}$ for the nest $\{p_j: j \in \mathbf{Z}\}$ and $\text{alg } \mathcal{L}$ for the associated nest algebra $\{T \in B(H): (1 - p_j)Tp_j = 0 \text{ for all } j\}$. It follows from Arveson's result [2, Theorem 3.3] that, in this case, every invertible operator T in $B(H)$ can be factored with respect to $(B(H), \text{alg } \mathcal{L})$. Let $M \subseteq B(L^2(\mathbf{T}))$ be the algebra of all the operators of multiplication by L^∞ functions and let T be an invertible operator in M . Since $M \cap \text{alg } \mathcal{L}$ is the algebra of all the operators of multiplication by H^∞ functions, we know from the classical theory of $H^\infty(\mathbf{T})$ that, in fact, T can be factored with respect to $(M, M \cap \text{alg } \mathcal{L})$.

It is natural to ask whether this situation is more general. To state it more precisely, let $\mathcal{L}$ be a lattice of projections in $B(H)$, let T be an invertible operator that can be factored with respect to $(B(H), \text{alg } \mathcal{L})$ (where $\text{alg } \mathcal{L} = \{T \in B(H): (1 - P)TP = 0 \text{ for all } P \text{ in } \mathcal{L}\}$) and let M be a von Neumann algebra that contains T . Can T be factored with respect to $(M, M \cap \text{alg } \mathcal{L})$?

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In this paper we prove that if $\mathcal{L}$ is a subspace lattice associated with a unitary representation $\{W_t: t \in A\}$ of a compact abelian group A with an ordered dual group Γ and if M is invariant under $\text{ad } W_t$ (i.e. $W_t M W_t^* = M$) for all $t \in A$, then whenever T can be factored with respect to $(B(H), \text{alg } \mathcal{L})$, it can be factored with respect to $(M, M \cap \text{alg } \mathcal{L})$.

As corollaries we derive results about factorization with respect to CSL subalgebras and analytic operator algebras.

For more about the factorization problem and other related problems see [1, 2, 3, 5, 6, 8, 9, 11].

We will assume that all Hilbert spaces are complex and separable and $B(H)$ will denote the algebra of all bounded operators on the Hilbert space H . If $S \subseteq H$ is a set of vectors, $[S]$ will denote the closed subspace spanned by S .

2. The factorization results. Let A be a compact abelian group with dual Γ . Let $W = \{W_t: t \in A\}$ be a strongly continuous unitary representation of A on a Hilbert space H . By Stone's Theorem, there is a unique projection valued measure $P(\cdot)$ on the Borel sets of Γ such that

$$W_t = \int_{\Gamma} \langle t, p \rangle dP(p), \quad t \in A,$$

where $\langle t, p \rangle$ is the dual pairing of A and Γ . Let $\Sigma \subseteq \Gamma$ be a positive semigroup; i.e. $\Sigma + \Sigma \subseteq \Sigma$ and $\Sigma \cap (-\Sigma) = \{0\}$.

A lattice of projections in $B(H)$ is called a *subspace lattice* if it contains 0 and I and is closed in the strong operator topology. Write $\mathcal{L}$ for the (commutative) subspace lattice generated by $\{P(q + \Sigma): q \in \Gamma\}$. Also let α_t , $t \in A$, be the automorphism implemented by W_t , i.e. $\alpha_t(T) = W_t T W_t^*$, $T \in B(H)$.

THEOREM. Suppose $T = UA$ where U is a unitary operator in $B(H)$ and A lies in $\text{alg } \mathcal{L} \cap (\text{alg } \mathcal{L})^{-1}$.

Then $T = WB$ where W is a unitary operator, B lies in $(\text{alg } \mathcal{L}) \cap (\text{alg } \mathcal{L})^{-1}$ and both W and B lie in the smallest $\{\alpha_t\}$ -invariant von Neumann algebra that contains T .

PROOF. We have $T = UA$, $A \in \text{alg } \mathcal{L} \cap (\text{alg } \mathcal{L})^{-1}$ and U is a unitary operator in $B(H)$. Write M for the smallest $\{\alpha_t\}$ -invariant von Neumann algebra that contains T . For $q \in \Gamma$, $[AP(q + \Sigma)H] = [P(q + \Sigma)H]$; hence $[TP(q + \Sigma)H] = U[P(q + \Sigma)H]$. Since $\alpha_t(M) = M$, $t \in A$, we have $\alpha_t(M') = M'$; so that α defines a compact action on M' . Write $R_q = \{S \in M': \alpha_t(S) = \langle t, q \rangle S \text{ for every } t \in A\}$, $q \in \Gamma$. It is known that the σ -weakly closed span of $\bigcup_{q \in \Gamma} R_q$ is M' . Also, if $S \in R_q$ and $S = V|S|$ is its polar decomposition then $V \in R_q$ and $|S| \in R_0$. Now fix $q \in \Gamma$ and a partial isometry V in R_q . Then

$$W_t V W_t^* = \langle t, q \rangle V, \quad t \in A.$$

Then $V^* W_t V = \langle t, q \rangle V^* V W_t$; hence

$$\int_{\Gamma} \langle t, p \rangle d(V^* P(\cdot) V)(p) = \int_{\Gamma} \langle t, p + q \rangle V^* V dP(p) = \int_{\Gamma} \langle t, p \rangle V^* V dP(p - q).$$

Therefore,

$$V^*P(\gamma + \Sigma)V = V^*VP(\gamma - q + \Sigma), \quad \gamma \in \Gamma.$$

Let $Q(\cdot)$ be the projection valued measure defined by $Q(\cdot) = UP(\cdot)U^*$ and write $U_t = \int_{\Gamma} \langle t, p \rangle dQ(p) = UW_tU^*$, $t \in A$. We have

$$\begin{aligned} V^*[Q(\gamma + \Sigma)H] &= V^*[UP(\gamma + \Sigma)H] = V^*[TP(\gamma + \Sigma)H] \\ &= [TV^*P(\gamma + \Sigma)H] = [TV^*VP(\gamma - q + \Sigma)H] \\ &= V^*V[Q(\gamma - q + \Sigma)H]. \end{aligned}$$

Hence $V^*Q(\gamma + \Sigma)V = V^*VQ(\gamma - q + \Sigma)$, $\gamma \in \Gamma$.

Integrating, we get for $t \in A$,

$$\begin{aligned} V^*U_tV &= \int \langle t, p \rangle d(V^*Q(p)V)(p) = \int \langle t, p \rangle V^*V dQ(p - q) \\ &= \int \langle t, p \rangle \langle t, q \rangle V^*V dQ(p) = \langle t, q \rangle V^*VU_t. \end{aligned}$$

For $q = 0$ and V a unitary operator in R_0 ($= \{S \in M': \alpha_t(S) = S, t \in A\}$) we get $V^*U_tV = U_t$; hence $U_t \in R'_0$. For a partial isometry V in R_q , $q \in \Gamma$, we now have

$$U_tVU_t^* = U_tVV^*VU_t^* = VV^*U_tVU_t^* = \langle t, q \rangle VV^*V = \langle t, q \rangle V = \alpha_t(V).$$

Hence, for every $S \in M'$, $U_tSU_t^* = \alpha_t(S) = W_tSW_t^*$.

Now write

$$\tilde{W}_t = \begin{pmatrix} W_t & 0 \\ 0 & U_t \end{pmatrix} \in B(H \oplus H),$$

and

$$\tilde{R} = \left\{ \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} : S \in M' \right\} \subseteq B(H \oplus H).$$

From the above it follows that $\tilde{W}_t\tilde{R}\tilde{W}_t^* = \tilde{R}$. Hence also $\tilde{W}_t(\tilde{R})'\tilde{W}_t^* = \tilde{R}'$. But

$$\tilde{R}' = \left\{ \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} : S_{ij} \in M \right\}.$$

Let γ_t be $\text{ad } \tilde{W}_t$; i.e., $\gamma_t(S) = \tilde{W}_tS\tilde{W}_t^*$ for $S \in B(H \oplus H)$ and write $\varepsilon(S) = \int_A \gamma_t(S) dt$. Then ε is a faithful normal expectation from $B(H \oplus H)$ onto $\{S \in B(H \oplus H) : \gamma_t(S) = S, t \in A\}$ and $\varepsilon(\tilde{R}') = \{S \in \tilde{R}' : \gamma_t(S) = S, t \in A\}$. We have $\tilde{W}_t = \int_{\Gamma} \langle t, p \rangle d\tilde{P}(p)$ where

$$\tilde{P}(E) = \begin{pmatrix} P(E) & 0 \\ 0 & Q(E) \end{pmatrix}$$

and if we write $\tilde{\mathcal{L}}$ for the lattice

$$\left\{ \begin{pmatrix} P(q + \Sigma) & 0 \\ 0 & Q(q + \Sigma) \end{pmatrix} : q \in \Gamma \right\}$$

then $\text{alg } \tilde{\mathcal{L}} = \{S \in B(H \oplus H) : \text{sp}_{\gamma}(S) \subseteq \Sigma\}$ where $\text{sp}_{\gamma}(\cdot)$ is Arveson's spectrum (see [4, Proposition 2.1]).

Also note that ε , restricted to $\text{alg } \tilde{\mathcal{L}}$, is multiplicative.

Now write

$$\tilde{T} = \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \in \tilde{R}'.$$

Since $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ is in the range of ε and ε is an expectation we get $\varepsilon(\tilde{T}) = \begin{pmatrix} 0 & T_1 \\ T_0 & 0 \end{pmatrix}$ for some T_0, T_1 in M . Since $[TP(q + \Sigma)H] = U[P(q + \Sigma)H] = [Q(q + \Sigma)H]$ and $[T^{-1}Q(q + \Sigma)H] = [A^{-1}U^*Q(q + \Sigma)H] = [A^{-1}P(q + \Sigma)H] = [P(q + \Sigma)H]$, $\tilde{T} \in \text{alg } \tilde{\mathcal{L}}$. Since $\tilde{T}^2 = I$ and ε is multiplicative on $\text{alg } \tilde{\mathcal{L}}$,

$$\begin{pmatrix} 0 & T_1 \\ T_0 & 0 \end{pmatrix}^2 = I;$$

hence T_0 is invertible in M . Let $T_0 = W|T_0|$ be its polar decomposition. Then W is a unitary operator in M . Since $\varepsilon(\tilde{T}) \in \tilde{\mathcal{L}}'$ we get $T_0P(q + \Sigma) = Q(q + \Sigma)T_0$ for every $q \in \Gamma$. Hence

$$\begin{aligned} T_0^*T_0P(q + \Sigma) &= T_0^*Q(q + \Sigma)T_0 = (Q(q + \Sigma)T_0)^*T_0 \\ &= (T_0P(q + \Sigma))^*T_0 = P(q + \Sigma)T_0^*T_0; \end{aligned}$$

so that $|T_0| \in \mathcal{L}'$ and $WP(q + \Sigma)W^* = Q(q + \Sigma)$, $q \in \Gamma$. Hence $B = W^*T$ satisfies $[BP(q + \Sigma)H] = W^*[TP(q + \Sigma)H] = [P(q + \Sigma)H]$ and, consequently, $B \in \text{alg } \mathcal{L} \cap (\text{alg } \mathcal{L})^{-1} \cap M$. $\square$

COROLLARY 1. *Let $\mathcal{L}$ be a commutative subspace lattice contained in a von Neumann algebra $M \subseteq B(H)$ and having a totally atomic core (that is, the von Neumann algebra generated by $\mathcal{L}$ is totally atomic). If $T \in M$ and $T = UA$ where $U \in B(H)$ is unitary and $A \in \text{alg } \mathcal{L} \cap (\text{alg } \mathcal{L})^{-1}$ then U and A can be chosen in M .*

PROOF. Using [4, Theorem 4.4] there is a compact abelian group A , a positive semigroup Σ in $\Gamma = \hat{A}$ and a strongly continuous unitary representation $W = \{W_t\}_{t \in A}$ on H with $W_t \in M$ such that $\mathcal{L}$ is generated by $\{P(q + \Sigma): q \in \Gamma\}$, where $P(\cdot)$ is the projection valued measure associated with $\{W_t\}$. The corollary now follows immediately from the theorem. $\square$

Note that when $\mathcal{L}$ is a well-ordered nest this result was obtained in [9, Corollary 2.4].

With Γ and Σ as above Γ is an ordered group where $p \leq q$ if and only if $q - p \in \Sigma$ and $p < q$ if $p \leq q$, $p \neq q$. If $\Sigma \cup (-\Sigma) = \Gamma$ then the order is linear. We say that the (linearly) ordered group (Γ, Σ) is *scattered* if it does not contain any dense subset. (Here $\Lambda \subseteq \Gamma$ is *dense* in Γ if for every $p < q$, p, q in Γ , there is $\gamma \in \Lambda$ such that $p < \gamma < q$.) If $\Gamma = \mathbf{Z}^n$ and $\Sigma = \{(m_1, m_2, \dots, m_n) = \bar{m}: \text{for some } k < n, m_1 = \dots = m_k = 0 \text{ and } m_{k+1} > 0 \text{ or } \bar{m} = (0, \dots, 0)\}$ then (Γ, Σ) is scattered. In fact, every scattered countable ordered group is $\mathbf{Z}^\gamma$ where γ is a countable ordinal (see [10, Theorem 8.5]). If (Γ, Σ) is scattered then its completion (as a lattice) is countable (see [10, Exercise 5.33]).

Let $\alpha: A \rightarrow \text{Aut}(M)$ be a σ -weakly continuous representation of A as a group of $*$ -automorphisms on a von Neumann algebra M . If Λ is a (closed) subset of Γ we write $M^\alpha(\Lambda) = \{T \in M: \text{sp}_\alpha(T) \subseteq \Lambda\}$ where $\text{sp}_\alpha(\cdot)$ is Arveson's spectrum. If $\Sigma \subseteq \Gamma$ is a positive semigroup then [7] $M(\Sigma)^\alpha$ is a σ -weakly closed algebra and we will refer to it as the *analytic subalgebra* of M associated with α and Σ . We have $M^\alpha(\Sigma) \cap M^\alpha(\Sigma)^* = \{T \in M: \alpha_t(T) = T \text{ for all } t \in A\}$.

In [11, Theorem 3.10] it is shown that when $A = \mathbf{T}$ (the unit circle) and $\Sigma = \{n \in \mathbf{Z}: n \geq 0\}$ then every $T \in M$ can be factored with respect to $(M, M^\alpha(\Sigma))$. The following corollary extends this result.

COROLLARY 2. *Let M be a σ -finite von Neumann algebra and A be a compact abelian group whose dual, Γ , is a scattered linearly ordered group with a positive semigroup Σ . Let α be a σ -weakly continuous representation of A as a group of $*$ -automorphisms of M . Then every invertible operator T in M can be written as UA where U is a unitary operator in M and $A \in M^\alpha(\Sigma) \cap M^\alpha(\Sigma)^{-1}$.*

PROOF. Since A is compact there is a faithful normal expectation ε_0 from M onto the fixed point algebra $M^\alpha (= \{a \in M: \alpha_t(a) = a, t \in A\})$; namely $\varepsilon_0(a) = \int_A \alpha_t(a) dt$. Therefore there exists a faithful normal $\{\alpha_t\}$ -invariant state ϕ on M . Considering the Gelfand-Naimark-Segal construction of ϕ we may suppose that M has a separating and cyclic vector $\xi_0 \in H$ such that $\phi(x) = \langle x\xi_0, \xi_0 \rangle$, $x \in M$. Since $\phi \circ \alpha_t = \phi$, $t \in A$, the map $a\xi_0 \rightarrow \alpha_t(a)\xi_0$, $a \in M$, defines a unitary operator W_t on H . Clearly $\{W_t\}$ is a strongly continuous unitary representation of A on H and $\alpha_t(a) = W_t a W_t^*$. By [4, Proposition 2.1], $M \cap \text{alg } \mathcal{L} = M^\alpha(\Sigma)$ where $\mathcal{L}$ is the subspace lattice generated by $\{P(q + \Sigma): q \in \Gamma\}$ (and $P(\cdot)$ is the projection valued measure associated with $\{W_t\}$). Since Γ is totally ordered, $\mathcal{L}$ is a nest and, since Γ is scattered, $\mathcal{L}$ is countable (and complete).

Using [5, Proposition 4.1] we find that every invertible operator $T \in B(H)$ can be written as UA where U is a unitary operator and $A \in \text{alg } \mathcal{L} \cap (\text{alg } \mathcal{L})^{-1}$. Using the theorem we can choose A, U in M , provided $T \in M$. $\square$

For the next corollary let α be an action of $A = \mathbf{T}^n$ on a σ -finite von Neumann algebra M . As in the proof of Corollary 2, we can represent M on a Hilbert space H such that there is a vector ξ_0 in H that is a separating and cyclic vector for M , and for $a \in M$, $t \in A$, we have $\langle a\xi_0, \xi_0 \rangle = \langle \alpha_t(a)\xi_0, \xi_0 \rangle$.

COROLLARY 3. *Let α , H and M be as above and let $\Sigma = \{(m_1, m_2, \dots, m_n) \in \mathbf{Z}^n: m_i \geq 0\}$. Then, for an invertible operator $T \in M$, T can be factored as UA (where U is a unitary operator and $A \in M^\alpha(\Sigma) \cap M^\alpha(\Sigma)^{-1}$) if and only if the projections onto the subspaces $\{[TM^\alpha(p + \Sigma)\xi_0]: p \in \Gamma\}$ pairwise commute.*

PROOF. Note that $[M^\alpha(p + \Sigma)\xi_0] = [P(p + \Sigma)H]$ where $P(\cdot)$ is the projection valued measure of $\{W_t\}$ and W_t is defined by $W_t a \xi_0 = \alpha_t(a)\xi_0$, $a \in M$.

The proof now follows from [3, Theorem 4] and the theorem. $\square$

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