A HILBERT CUBE L-S CATEGORY

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ABSTRACT. Let M be a compact connected Hilbert cube manifold (Q -manifold). Define $C_z(M)$ to be the smallest integer k such that M can be covered with k open subsets each of which is homeomorphic to $Q \times [0, 1)$. Recently L. Montejano proved that, for every compact connected polyhedron P , $C_z(P \times Q) = \text{cat}(P) + 1$, where $\text{cat}(P)$ is the Lusternik-Schnirelmann category of P . Using a different approach, we prove a noncompact analog of the above theorem by showing that $C_z(P \times Q \times [0, 1)) = \text{cat}(P)$ for every compact connected polyhedron P .

1. Introduction. For a compact polyhedron P , the Lusternik-Schnirelmann (L-S) category of P , $\text{cat}(P)$, is the smallest integer k such that P can be covered with k subpolyhedra each of which is null homotopic in P . A Q -manifold is a separable metric space modeled on the Hilbert cube $Q = [0, 1]^\infty$. Let M be a connected Q -manifold. Define $C_z(M)$ to be the smallest integer k such that M can be covered with k open subsets each of which is homeomorphic to $Q \times [0, 1)$. Recently, L. Montejano proved the following theorem concerning the relationship between $\text{cat}(P)$ and $C_z(P \times Q)$ ($P \times Q$ being a Q -manifold [WE]):

THEOREM [MO₁]. *For every compact connected polyhedron P ,*

$$C_z(P \times Q) = \text{cat}(P) + 1.$$

Obviously, $C_z(Q) = 2$. On the other hand, suppose that M is a compact connected Q -manifold with $C_z(M) = 2$. By the triangulation theorem of [CH₁], M is homeomorphic to $P \times Q$ for some compact connected polyhedron P . Hence $\text{cat}(P) = 1$ by the above theorem, i.e., P is contractible. Thus M is homeomorphic to Q , since the Hilbert cube is the only contractible compact Q -manifold [CH₁].

Extending to the noncompact case, we observe that if $P = \{\text{point}\}$,

$$C_z(P \times Q \times [0, 1)) = 1 = \text{cat}(P).$$

Hence it is natural to ask if $C_z(P \times Q \times [0, 1)) = \text{cat}(P)$ in general. In this note we answer the question affirmatively.

THEOREM 1. *For every compact polyhedron P ,*

$$C_z(P \times Q \times [0, 1)) = \text{cat}(P).$$

For an arbitrary space X , let $\text{cat}'(X)$ denote the smallest integer k such that X can be covered by k open sets each of which is null-homotopic in X . For a

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compact polyhedron P , it is easily seen that $\text{cat}(P) = \text{cat}'(P) = \text{cat}'(P \times Q)$. This observation, together with the triangulation theorem referred to above, shows that Montejano's result and our result may be formulated as follows:

COROLLARY. *For every compact connected Q -manifold M , $C_z(M) = \text{cat}'(M) + 1$ and $C_z(M \times [0, 1)) = \text{cat}'(M)$.*

PROOF OF THEOREM 1. It is easy to verify that $\text{cat}(P) \leq C_z(P \times Q \times [0, 1))$. On the other hand, suppose that $\text{cat}(P) = k$. Let $P_1, P_2, \dots, P_k$ be a null-homotopic cover of P . Denote $\tilde{M} = P \times Q \times \{0\} \subset P \times Q \times [0, 1) = \tilde{M}$. Let $B_i = P_i \times Q \times \{0\}$. B_i is a null-homotopic $\mathbf{Z}$ -set of $\tilde{M}$. By standard I-D topology we may assume that the cone of B_i is imbedded in $\tilde{M}$ as a $\mathbf{Z}$ -set in such a way that the base coincides with B_i . Since the cone contracts to its vertex v_i and v_i has an open neighborhood U_i homeomorphic to $Q \times [0, 1)$, by $\mathbf{Z}$ -set unknotting there is a homeomorphism on $\tilde{M}$ taking B_i into U_i . So, without loss of generality we may assume that $B_i \subset U_i$. Let $\{Q_n | n = 1, 2, \dots\}$ denote the countable collection of all end-faces of Q ; that is, each $Q_n = \pi_i^{-1}(0)$ or $\pi_i^{-1}(1)$ for some i , where π_i is the i th-projection of Q into the i th-factor. Let $A_{nm} = P_i \times Q_n \times [0, m/(m+1)]$, where $n, m = 1, 2, \dots$, and let $A_i = P_i \times Q \times [0, 1)$.

Each A_{nm} is a $\mathbf{Z}$ -set in $\tilde{M}$ and is contractible to $P_i \times Q_n \times \{0\} \subset U_i$. We want to employ the engulfing apparatus of $[\mathbf{CH}_1]$ to construct an open imbedding h_i of $Q \times [0, 1)$ into $\tilde{M}$ such that $\bigcup_{n,m} A_{nm} \subset h_i(Q \times [0, 1))$. Let $g_i: Q \times [0, 1) \rightarrow U_i$ be a homeomorphism. The key is to verify the following:

LEMMA A. *For any $\mathbf{Z}$ -set $K \subset P_i \times Q \times [0, r]$, $r < 1$, there is an open imbedding $g'_i: Q \times [0, 1) \rightarrow \tilde{M}$ such that*

$$g'_i|(Q \times [0, \frac{1}{2}]) = g_i|(Q \times [0, \frac{1}{2}]) \quad \text{and} \quad g'_i(Q \times [0, \frac{2}{3})) \supset K.$$

PROOF OF LEMMA A. Let $K' = K - g_i(Q \times [0, \frac{1}{2}))$, and assume $K' \neq \emptyset$. Since $P_i \times Q \times \{0\} \subset U_i$, K' is contractible to a subset of $g_i(Q \times [0, \frac{1}{2}))$. Using the fact that $Q \times [0, 1)$ is contractible to $Q \times [\frac{1}{2}, 1)$, we may assume that K' is homotopic to a subset of $g_i(Q \times [\frac{1}{2}, \frac{2}{3}])$ and the homotopy lies in the Q -manifold $N = \tilde{M} - g_i(Q \times [0, \frac{1}{2}))$. Since $g_i(Q \times \{\frac{1}{2}\})$ is a Hilbert cube and a $\mathbf{Z}$ -set in the Hilbert cube $g_i(Q \times [\frac{1}{2}, \frac{2}{3}]) \subset N$, using $\mathbf{Z}$ -set unknotting we may replace the homotopy by another homotopy $\{\phi_t\}$ of K' into N so that $\phi_0 = \text{identity}$, $\phi_t|K' \cap g_i(Q \times \{\frac{1}{2}\}) = \text{identity}$ for all t , $\phi_1(K') \subset g_i(Q \times [\frac{1}{2}, \frac{2}{3}])$ and for any $x \in K' - g_i(Q \times \{\frac{1}{2}\})$, $\{\phi_t(x)\}_t \subset N - g_i(Q \times \{\frac{1}{2}\})$. By $\mathbf{Z}$ -set unknotting there is a homeomorphism $f: N \rightarrow N$ such that $f|g_i(Q \times \{\frac{1}{2}\}) = \text{identity}$ and $f(K') \subset g_i(Q \times [\frac{1}{2}, \frac{2}{3}))$. Define g'_i by $g'_i|(Q \times [0, \frac{1}{2}]) = g_i|(Q \times [0, \frac{1}{2}])$, $g'_i|(Q \times [\frac{1}{2}, 1)) = f^{-1}g_i$. g'_i is what we wanted.

By Lemma A and the engulfing lemma of $[\mathbf{CH}_1]$ we can construct an open imbedding $h_i: Q \times [0, 1) \rightarrow \tilde{M}$ such that $\bigcup_{n,m} A_{nm} \subset h_i(Q \times [0, 1))$. Let $V_i = h_i(Q \times [0, 1))$ and $V = \bigcup_{i=1}^k V_i$. Then V is an open set in $\tilde{M}$ containing the cap-set $P \times \bigcup_{n=1}^\infty Q_n \times [0, 1)$; thus the complement $\tilde{M} - V$ is a $\mathbf{Z}$ -set $[\mathbf{CH}_2]$. It follows that V and $\tilde{M}$ are homotopically equivalent, and therefore the manifold $V \times [0, 1)$ and $\tilde{M}$ are homeomorphic. Since each $V_i \times [0, 1)$ is homeomorphic to $Q \times [0, 1)$, $C_z(\tilde{M}) = C_z(V \times [0, 1)) \leq k$.

The theory extends to the following noncompact case.

THEOREM 2. *For every connected Q -manifold M ,*

$$C_z(M \times [0, 1)) = \text{cat}'(M).$$

PROOF. Since $\text{cat}'(M)$ is a homotopy invariant, $\text{cat}'(M) = \text{cat}'(M \times [0, 1)) \leq C_z(M \times [0, 1))$. On the other hand, suppose $\text{cat}'(M) = k$. It can be easily shown that M has a null-homotopic closed cover $C_1, C_2, \dots, C_k$. Consider $\tilde{M} = M \times \{0\} \subset M \times [0, 1)$. As in the proof of Theorem A, each $C_i \subset U_i$ where U_i is an open set homeomorphic to $Q \times [0, 1)$. Let $g_i: Q \times [0, 1) \rightarrow U_i$ be a homeomorphism. Denote $A_{nm} = C_i \times Q_n \times [0, m/(m+1)]$. By the same argument as in the proof of Lemma A, there is an open imbedding $g'_i: Q \times [0, 1) \rightarrow \tilde{M}$ such that

$$g'_i|_{(Q \times [0, \frac{1}{2}])} = g_i|_{(Q \times [0, \frac{1}{2}])} \quad \text{and} \quad g'_i(Q \times [0, \frac{2}{3})) \supset A_{nm}.$$

Following the same argument as that of Theorem 1, we conclude that $M \times [0, 1)$ is covered by k open sets each of which is homeomorphic to $Q \times [0, 1)$. Hence $C_z(M \times [0, 1)) \leq k$.

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