THE SET OF BALANCED POINTS WITH RESPECT TO S^1 AND S^3 ACTIONS OF MAPS INTO BANACH SPACE

NEŽA MRAMOR-KOSTA

(Communicated by Haynes R. Miller)

ABSTRACT. Let G be the group of units in the field F , which is either R , C or H , let X be a free G -space, and let f be a map from X to a Banach space E over F . In this paper we give an estimate for the size of the subset of X consisting of points at which the average of f is equal to zero. The result represents an extension of the Borsuk-Ulam-Yang theorem.

Let F be one of the fields R , C or H , and $G \subset F$ the unit sphere equipped with the standard group structure. Take any space X with a free action of G , let V be a finite-dimensional vector space over F with the standard action of G (multiplication by units), and let $f: X \rightarrow V$ be a continuous map. The size of the set of points in X at which the average of f is zero can be described in terms of an invariant called the index [5]. We would like to show that for a certain class of maps, this can be generalized to the case where the representation space V is replaced by an arbitrary Banach space E over F . This has already been done by Spannier and Holm [4] for the field R , that is, if X is a space with a free action of $G = Z_2$.

The *average* of a map $f: X \rightarrow E$, from a space X with a free action of G to a representation space E for G (possibly infinite-dimensional), is the map $\text{Av } f: X \rightarrow E$, defined by $\text{Av } f(x) = \int_G g^{-1} f(gx) dg$, where $\int$ denotes the Haar integral on G . The point $x \in X$ is a *balanced point* of f if $\text{Av } f(x) = 0$. The set of balanced points is denoted by $A(f)$.

PROPOSITION 1. *Let $f: X \rightarrow E$ be continuous.*

- (a) *$\text{Av } f$ is an equivariant continuous map.*
- (b) *If f is equivariant, then $\text{Av } f = f$.*
- (c) *For any map f , $A(f)$ is a closed invariant subset of X and $A(f) = A(\text{Av } f)$.*

PROOF. (a) $\text{Av } f$ is equivariant, since for any $h \in G$,

$$\text{Av } f(hx) = \int_G g^{-1} f(ghx) dg = h \int_G (gh)^{-1} f(ghx) d(gh) = h \text{Av } f(x).$$

The map $F: G \times X \rightarrow E$, defined by $F(g, x) = f(gx)$, is continuous. Thus, for any $\varepsilon > 0$ and for any pair $(g, x) \in G \times X$, there exist open neighborhoods $U_{g,x}$ of g and $V_{g,x}$ of x such that $\|f(gx) - f(hy)\| < \varepsilon$ for any $h \in U_{g,x}$ and $y \in V_{g,x}$.

Received by the editors November 24, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55M20, 55M40; Secondary 47H09.

Key words and phrases. Average, equivariant map, compact map, characteristic class, index, coindex of equivariant map.

Fix a point $x \in X$. The sets $\{U_{g,x}, g \in G\}$ form an open covering of G . Let $\{U_{g_1,x}, \dots, U_{g_n,x}\}$ be a finite subcovering and define $V = \bigcap_{j=1}^n V_{g_j,x}$. Then V is an open neighborhood of x , and for any pair $(h,y) \in G \times V$ there is an index j , such that $(h,y) \in U_{g_j,x} \times V_{g_j,x}$. This implies that $\|f(hx) - f(hy)\| < \varepsilon$ for all $y \in V$ and for all $h \in G$, so

$$\|\text{Av } f(x) - \text{Av } f(y)\| \leq \int_G \|f(gx) - f(gy)\| dg < \varepsilon,$$

so $\text{Av } f(x)$ is continuous.

(b) follows directly from the definition of $\text{Av } f$.

(c) follows from (a), (b), and the fact that $A(f) = (\text{Av } f)^{-1}(0)$.

Consider the case $F = R$ and let E be a Banach space over R with the standard action of $G = \{-1, 1\} = Z_2$ and S the unit sphere in E . The average map $\text{Av } f$, associated to a map $f: S \rightarrow E$, is given by $\text{Av } f(x) = (f(x) - f(-x))/2$ and the set of balanced points $A(f)$ is $\{x \in X | f(x) = f(-x)\}$. For E finite-dimensional, the classical Borsuk-Ulam theorem asserts that $A(f)$ is nonempty if f maps into a subspace of E of nonzero codimension. This is, in general, not true when E is infinite-dimensional; a counterexample can easily be found even in the Hilbert space l_2 . But it is true in any Banach space if f is restricted to a certain class of maps. By a theorem of Granas [3], $A(f) \neq \emptyset$ if $f: S \rightarrow E$ is a compact perturbation of the inclusion with image in a subspace $E_k \subset E$ of codimension $k > 0$.

PROPOSITION 2. *If $K: X \rightarrow E$ is a compact map, $\text{Av } K$ is compact.*

PROOF. Assume first that K is a finite-dimensional compact map, so $\text{Im } K$ is contained in a subspace E^n of E , and there is a constant C such that $\|K(x)\| \leq C$ for every $x \in X$. Clearly $\text{Im } \text{Av } K \subset E^n$, since the integral is a linear operation. For any $x \in X$ we have the following estimate:

$$\|\text{Av } K(x)\| = \left\| \int_G g^{-1} K(gx) dg \right\| \leq \int_G \|g^{-1} K(gx)\| dg \leq C \int_G dg = C.$$

So $\text{Im } \text{Av } K$ is a bounded subset of a finite-dimensional subspace and is therefore contained in a compact subset of E .

In case K is a general compact map, it is enough to show that $\text{Im } \text{Av } K$ is a totally bounded subset of E . In other words, we have to show that for any $\varepsilon > 0$ there is a finite set $\{x_1, \dots, x_n\}$, such that $\text{Im } \text{Av } K$ is covered by balls B_i of radius ε with centers at the points $x_1, \dots, x_n$. Let K_σ be a finite-dimensional compact σ -approximation for K . Then

$$\|\text{Av } K(x) - \text{Av } K_\sigma(x)\| \leq \int_G \|g^{-1}(K - K_\sigma)(gx)\| dg < \sigma \int_G dg = \sigma.$$

Since K_σ is a finite-dimensional compact map, we have already seen that $\text{Av } K_\sigma$ is compact, and therefore $\text{Im } \text{Av } K_\sigma$ is totally bounded. Choose $\sigma < \varepsilon/3$ and let $\{x_1, \dots, x_n\} \subset E$ be such a finite set that $\text{Im } \text{Av } K_\sigma$ is covered by balls of radius $\varepsilon/3$ with centers at x_i . Then, for any x there exists such an x_i that

$$\begin{aligned} \|\text{Av } K(x) - \text{Av } K(x_i)\| &\leq \|\text{Av } K(x) - \text{Av } K_\sigma(x)\| + \|\text{Av } K_\sigma(x) - \text{Av } K_\sigma(x_i)\| \\ &\quad + \|\text{Av } K_\sigma(x_i) - \text{Av } K(x_i)\| \\ &< \sigma + \varepsilon/3 + \sigma < \varepsilon. \end{aligned}$$

This shows that $\text{Im } \text{Av } K$ is totally bounded, so $\text{Av } K$ is compact.

From now on, X will be a paracompact space with a free action of G . Let $\Phi: X \rightarrow E_G = S^\infty$ be a classifying map. Denote the generator of the cohomology ring of the projective space $P^\infty F$ by $u_F \in H^d(P^\infty F)$ (where d is the dimension of F over R), and the characteristic class of the action of G on X , $(\Phi/G)^* u_F \in H^d(X/G)$, by $u_F(X)$. The cohomology theory is chosen so that it has the continuity property (for example, Čech or Alexander-Spanier cohomology). The coefficients are Z_2 in case $F = R$ and Z otherwise.

The *index* (or *coindex* by some notations) of X for $G = Z_2$ was defined by Connor and Floyd [1] as $\text{Ind}_R X = \sup\{n | (u_R(X))^n \neq 0\}$. For $G = S^1$ or S^3 , we will use the following definition of index:

$$\text{Ind}_F X = \sup\{n | (u_F(X))^n \text{ is of infinite order in } H^{nd}(X/G)\}.$$

It is easily seen that $\text{Ind } X \leq \text{Ind } Y$ if there exists an equivariant map $f: X \rightarrow Y$.

EXAMPLE. Let X be the unit sphere $S^{nd-1} \subset F^n$ with the standard action of G . Then $u_F(X) \in H^d(P^{n-1}F)$ is the generator of the cohomology of the projective space $P^{n-1}F = X/G$, so $\text{Ind}_F X = n - 1$.

The following theorem is due to Jaworowski [5].

THEOREM 1. *Let G act freely on a space X and orthogonally and freely outside the origin on a finite-dimensional representation space V for G over F . Let $f: X \rightarrow V$ be a map. Then $\text{Ind}_F A(f) \geq \text{Ind}_F X - \dim_F V$.*

It follows directly that $A(f) \neq 0$ if $\text{Ind}_F X \geq \dim_F V$. In the special case $F = R$ and $X = S^{n-1} \subset R^n$, this gives a version of the classical Borsuk-Ulam theorem.

Spanier and Holm [4] have defined the coindex of an equivariant map $\varphi: X \rightarrow E$, where X is a space with a free action of Z_2 and E is a Banach space over R . This definition can be generalized to the cases when $G = S^1$ and E is a Banach space over C , or $G = S^3$ and E is a Banach space over H .

DEFINITION. the *coindex* of an equivariant map $\varphi: X \rightarrow E$ is defined by $\text{Coind}_F \varphi = \inf\{\text{Ind}_F \varphi^{-1}(V) - \dim_F V, V \leq E \text{ finite-dimensional}\}$.

EXAMPLE. The inclusion $i: S \rightarrow E$, which is equivariant with respect to the standard action of G , has coindex -1 .

Let $f: X \rightarrow E$ be a compact perturbation of a proper equivariant map φ ; that is, $f = \varphi + K$, where φ is proper and equivariant and K is compact.

PROPOSITION 3. (a) f is a proper map.

(b) $\text{Av } f$ is a proper map.

PROOF. (a) Let $A \subset E$ be compact and let B be the compact set $\{a - y | a \in A, y \in \text{Cl } K(X)\}$. Then $f^{-1}(A) = (\varphi + K)^{-1}(A) \subset \varphi^{-1}(B)$. Since φ is proper, this implies that $f^{-1}(A)$ is a closed subset of a compact set and therefore compact.

(b)

$$\begin{aligned} \text{Av } f(x) &= \int_G g^{-1} f(gx) dg = \int_G g^{-1} [\varphi(gx) + K(gx)] dg \\ &= \int_G g^{-1} \varphi(gx) dg + \int_G g^{-1} K(gx) dg = \text{Av } \varphi(x) + \text{Av } K(x). \end{aligned}$$

Since φ is equivariant, $\text{Av } \varphi = \varphi$. By Proposition 2, $\text{Av } K$ is a compact map, so $\text{Av } f$ is a compact perturbation of a proper map, and therefore proper by (a).

THEOREM 2. *Let $\varphi: X \rightarrow E$ be a proper equivariant map, and the map $f: X \rightarrow E$ a compact perturbation of φ with image in a subspace $E_k \subset E$ of codimension k . Then*

$$\text{Ind}_F A(f) \geq \text{Coind}_F \varphi + k.$$

LEMMA 1. *Given a closed neighborhood U of $A(f)$, there exists an $\varepsilon > 0$ such that $\| \text{Av } f(y) \| < \varepsilon$ implies $y \in U$.*

PROOF. Assume this is false. Then there exists a sequence of points $y_i \in X - U$ such that $\| \text{Av } f(y_i) \| < 1/i$. But $\text{Av } f$ is a proper map by Proposition 3, so the sequence $\{y_i\}$ contains a convergent subsequence converging to some point $y_0 \in X - \text{Int}(U) \subset X - A(f)$. On the other hand, $\text{Av } f(y_0) = 0$, so $y_0 \in A(f) \subset U$, which is a contradiction.

LEMMA 2. *Let U be a closed neighborhood of $A(f)$. There is a finite-dimensional compact map $K': X \rightarrow E$ such that the map $f' = \varphi + K'$ has image in E_k and $A(f') \subset U$.*

PROOF. Let $\pi: E \rightarrow E$ be the projection to a k -dimensional subspace $E^k \subset E$ with $\ker \pi = E_k$. Then $\pi \circ f = \pi \circ \varphi + \pi \circ K$ is zero. Let K_δ be a compact finite-dimensional δ -approximation for K , and define $K' = K_\delta + \pi \circ K - \pi \circ K_\delta$ and $f' = \varphi + K'$. Then K' is a finite-dimensional compact map and $\pi \circ f' = 0$, so $\text{Im } f' \subset E_k$. For all $x \in X$,

$$\|f'(x) - f(x)\| = \|K'(x) - K(x)\| = \|(1 - \pi)(K - K_\delta)(x)\| \leq \|1 - \pi\|\delta.$$

Let $y \in A(f')$. Then

$$\begin{aligned} \| \text{Av } f(y) \| &= \| \text{Av } f(y) - \text{Av } f'(y) \| = \left\| \int_G g^{-1}(f - f')(gy) dg \right\| \\ &\leq \int_G \|(f - f')(gy)\| dg \leq \|1 - \pi\|\delta. \end{aligned}$$

This implies that $\|f(y) - f'(y)\| < \varepsilon$ if δ is sufficiently small, so $y \in U$ by Lemma 1. This proves that $A(f') \subset U$.

PROOF OF THEOREM 2. Choose a closed invariant neighborhood U of $A(f)$ with $\text{Ind}_F A(f) = \text{Ind}_F U$ (such a neighborhood exists by the continuity of the cohomology theory), and K' as in Lemma 2. The inclusion $A(f') \rightarrow U$ is equivariant, so $\text{Ind}_F U \geq \text{Ind}_F A(f')$, and $\text{Ind}_F A(f) \geq \text{Ind}_F A(f')$. This shows that it suffices to prove the theorem in case K is a finite-dimensional compact map.

In this case, $\text{Im } K$ is contained in some finite-dimensional subspace, say E^m , of E . Let E^n be any finite-dimensional subspace of E containing $E^m + E^k$. Denote by f_n the restriction of f to $\varphi^{-1}(E^n)$. If $x \in A(f)$, then $\text{Av } f(x) = \varphi(x) + \text{Av } K(x) = 0$, so $\varphi(x) = -\text{Av } K(x) \in E^n$. This shows that $A(f) \subset \varphi^{-1}(E^n)$, so $A(f) = A(f_n)$. By Theorem 1, $\text{Ind}_F A(f_n) \geq \text{Ind}_F \varphi^{-1}(E^n) - (n - k)$, and so

$$\text{Ind}_F A(f) = \text{Ind}_F A(f_n) \geq \text{Coind}_F \varphi + k.$$

COROLLARY. *If $f: S \rightarrow E$ is a compact perturbation of the inclusion with image in a subspace E_k of E of codimension $k > 0$, then $\text{Ind}_F A(f) \geq k - 1$. It follows that $A(f) \neq \emptyset$.*

The theorem is true also for other actions of the group G on E if the following condition is satisfied: Every invariant finite-dimensional subspace $E^m \subset E$ has an inner product such that the action of G restricted to E^m is orthogonal.

I would like to thank my advisor Professor Jan Jaworowski for his helpful comments and for his continuous interest in my work.

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INSTITUTE FOR MATHEMATICS, PHYSICS AND MECHANICS OF THE EDVARD KARDELJ
UNIVERSITY OF LJUBLJANA, JADRANSKA 19, 61000 LJUBLJANA, YUGOSLAVIA