ANSWER TO A PROBLEM OF MILLER

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ABSTRACT. In this note we give an affirmative answer to a problem of S. Miller. Our method is very simple.

Let $w(z)$ be a regular function in unit disc $|z| < 1$ with $w(0) = 0$. Thanks to Schwarz' Lemma, it is easy to deduce $|w(z)| < 1$ if $|w(z) + zw'(z)| < 1$ ($|z| < 1$). In 1978, S. S. Miller and P. T. Mocanu [1] showed that it is true that $|w(z) + zw'(z) + z^2w''(z)| < 1$ implies $|w(z)| < 1$.

Naturally one would ask if it is true that $|w(z) + zw'(z) + \cdots + z^n w^{(n)}(z)| < 1$ implies $|w(z)| < 1$ ($|z| < 1$) for all positive integers n . It was put forward by Miller [2] in 1979. In this note we give an affirmative answer to it.¹

THEOREM. *Let $w(z)$ be a regular function in unit disc $|z| < 1$ with $w(0) = 0$ and satisfy the following condition*

$$(1) \quad |w(z) + zw'(z) + \cdots + z^n w^{(n)}(z)| < 1 \quad (\text{for some integer } n \geq 2)$$

then

$$(2) \quad |w(z)| < \sqrt{\sum_{k=1}^{\infty} \frac{1}{p_k^2}} < 0.554$$

where

$$(3) \quad p_k = \begin{cases} 1 + k + k(k-1) + \cdots + k!, & \text{when } k \leq n, \\ 1 + k + k(k-1) + \cdots + k(k-1) \cdots (k-n+1), & \text{when } k > n. \end{cases}$$

REMARK. As noted by Goldstein et al. [3], the best-possible bound in (2) is $\frac{1}{2}$, which is sharp for $w(z) = \frac{1}{2}z$.

PROOF. Let

$$w(z) = \sum_{k=1}^{\infty} a_k z^k,$$

then

$$w(z) + zw'(z) + \cdots + z^n w^{(n)}(z) = \sum_{k=1}^{\infty} p_k a_k z^k,$$

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¹After finishing this note, which had been presented at the 4th Chinese National Complex Function Theory Meeting held in August, 1984, I learned that M. Goldstein, R. R. Hall, J. Sheil-Small, and H. L. Smith, [3] have solved a considerably more general problem. Very recently, we came across a paper by Sakari Topilla [4], which also gives an affirmative answer to this problem, but I still think this note is interesting because of its simplicity.

where p_k is defined by (3). From (1) and Littlewood's subordination theorem we have

$$\sum_{k=1}^{\infty} p_k^2 |a_k|^2 \leq 1.$$

Making use of Cauchy's inequality we derive

$$(4) \quad |w(z)| < \sum_{k=1}^{\infty} |a_k| \leq \sqrt{\left(\sum_{k=1}^{\infty} \frac{1}{p_k^2}\right) \left(\sum_{k=1}^{\infty} p_k^2 |a_k|^2\right)} \leq \sqrt{\sum_{k=1}^{\infty} \frac{1}{p_k^2}}.$$

So the first part of (2) is obtained.

From the definition of p_k it is easy to see

$$p_k \geq 1 + k + k(k-1) = 1 + k^2.$$

Applying the integral estimate

$$\sum_{k=n+1}^{\infty} \frac{1}{(1+k^2)^2} < \int_n^{\infty} \frac{dx}{(1+x^2)^2} = \frac{1}{2} \left(\frac{\pi}{2} - \arctan n \right) - \frac{n}{2(n^2+1)}$$

to $\sum_{k=11}^{\infty} 1/p_k^2$, we obtain

$$\sum_{k=1}^{\infty} \frac{1}{p_k^2} \leq \sum_{k=1}^{10} \frac{1}{(1+k^2)^2} + \sum_{k=11}^{\infty} \frac{1}{(1+k^2)^2} < 0.3068827.$$

This inequality, combining with (4), gives

$$|w(z)| < \sqrt{\sum_{k=1}^{\infty} \frac{1}{p_k^2}} < 0.5539699.$$

The theorem is proved.

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