ON CONTINUED FRACTIONS OF GIVEN PERIOD

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ABSTRACT. A direct proof is given of the fact that, for any $k \in \mathbb{Z}^+$, there are infinitely many squarefree integers N , where the continued fraction expansion of $\sqrt{N}$ has period equal to k .

I. Introduction. In *Problems on periodic simple continued fractions* [1], Chowla and Chowla exhibit several conjectures concerning continued fraction expansions. In particular, one conjecture is the strong statement that, for any positive integer k , there exist infinitely many prime N with the continued fraction expansion of $\sqrt{N}$ having period k . Resolving this conjecture in the affirmative would show that, for example, in the case $k = 1$, there are infinitely many primes of the form $n^2 + 1$, $n \in \mathbb{Z}$, which is a longstanding unsolved problem.

It is, however, not difficult to show that, for any positive integer k , there exist infinitely many squarefree N with a continued fraction expansion of period k . This fact is a simple corollary of a more general result to be proved in this paper.

Let the continued fraction expansion of $\sqrt{N}$, N a nonsquare integer, be written $\sqrt{N} = (a_0; \overline{a_1, a_2, \dots, a_{k-1}, a_k = 2a_0})$, where k is the period and the a_i ($0 \leq i \leq k$) are the partial denominators. This is the same as the following:

$$\begin{array}{r} \sqrt{N} = a_0 + 1 \\ \hline a_1 + 1 \\ \hline a_2 + 1 \\ \hline a_3 + \cdots \\ \hline \cdots \\ \hline \cdots + 1 \\ \hline a_{k-2} + 1 \\ \hline a_{k-1} + 1 \\ \hline \sqrt{N} + a_0 \end{array}$$

Fix $k \in \mathbb{Z}^+$. Given a set of positive integers $\{a_n\}_{n=0,1,\dots,k-1}$ we define

$$\begin{aligned} P_{-1} &= 1, & Q_{-1} &= 0, \\ P_0 &= a_0, & Q_0 &= 1, \\ P_n &= a_n P_{n-1} + P_{n-2}, & Q_n &= a_n Q_{n-1} + Q_{n-2} \quad \text{for } n = 1, 2, \dots, k-1. \end{aligned}$$

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II. Results.

THEOREM. *Let $[N]$ denote the greatest integer $\leq \sqrt{N}$. Then the equation*

$$\sqrt{N} = ([\sqrt{N}]; \overline{a_1, a_2, \dots, a_{k-2} = a_2, a_{k-1} = a_1, a_k = 2[\sqrt{N}]})$$

has, for any symmetric set of positive integers $\{a_1, \dots, a_{k-1}\}$, infinitely many squarefree solutions N whenever either Q_{k-2} or $(Q_{k-2}^2 - (-1)^k)/Q_{k-1}$ is even. If both quantities are odd, then there are no solutions N even if the squarefree condition is dropped.

COROLLARY. *For any positive integer k there exist infinitely many squarefree integers N with a continued fraction expansion of period k .*

Proof of the Theorem and Corollary will follow as a result of three lemmas that are set out below. To simplify the presentation it will be assumed that $k > 1$. The special case $k = 1$ will be treated separately.

LEMMA 1. *With P_n and Q_n defined as before we have*

$$\begin{aligned} & \frac{a_0 + 1}{\frac{a_1 + 1}{\frac{a_2 + \dots}{\dots}}} = \frac{xP_{k-1} + P_{k-2}}{xQ_{k-1} + Q_{k-2}} \quad \text{where } x \in \mathbf{R}^+, \\ & \dots + 1 \over \frac{a_{k-1} + 1}{x} \end{aligned}$$

as well as the identity

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n+1} \quad \text{for } n = 0, 1, \dots, k-1.$$

PROOF. The proof of Lemma 1 is by induction and appears in standard texts on continued fractions. See, for example, [2].

Consider the following formula, which we arrive at by setting $x = \sqrt{N} + a_0$ in the equation in Lemma 1:

$$\begin{aligned} & \frac{a_0 + 1}{\frac{a_1 + 1}{\frac{a_2 + \dots}{\dots}}} = \frac{(\sqrt{N} + a_0)P_{k-1} + P_{k-2}}{(\sqrt{N} + a_0)Q_{k-1} + Q_{k-2}} \\ & \dots + 1 \over \frac{a_{k-1} + 1}{\sqrt{N} + a_0} \end{aligned}$$

For the left-hand side of the above formula to equal $\sqrt{N}$ (ensuring that $\sqrt{N} = (a_0; \overline{a_1, a_2, \dots, a_{k-1}, a_k = 2a_0})$) with N a nonsquare integer, we equate both the

integer parts and coefficients of $\sqrt{N}$ in the equation obtained by setting the right-hand side equal to $\sqrt{N}$. This gives

$$P_{k-1} = a_0 Q_{k-1} + Q_{k-2} \quad \text{and} \quad N Q_{k-1} = a_0 P_{k-1} + P_{k-2}.$$

From the identity in Lemma 1, with $n = k - 1$, we get

$$P_{k-2} = (P_{k-1} Q_{k-2} - (-1)^k) / Q_{k-1}.$$

These three equations give us, as a sufficient condition for N to be an integer with $\sqrt{N}$ having a continued fraction expansion with period $= k$, the following:

$$(*) \quad (N - a_0^2) Q_{k-1} - (2a_0) Q_{k-2} = (Q_{k-2}^2 - (-1)^k) / Q_{k-1}.$$

A simple induction with the defining formula for the Q_n 's shows that Q_{k-1} and Q_{k-2} are coprime. Hence the integer solutions x and y of

$$x Q_{k-1} - y Q_{k-2} = z, \quad \text{with } z \text{ any integer,}$$

are exactly those of the form

$$\begin{aligned} x &= (-1)^{k+1} z (Q_{k-2}^2 - (-1)^k) / Q_{k-1} + m Q_{k-2} \quad \text{and} \\ y &= (-1)^{k+1} z Q_{k-2} + m Q_{k-1}, \quad \text{where } m \text{ is any integer.} \end{aligned}$$

So, $(*)$ is equivalent to the following:

$$\begin{aligned} N - a_0^2 &= (-1)^{k+1} (Q_{k-2}^2 - (-1)^k)^2 / Q_{k-1}^2 + m Q_{k-2} \quad \text{and} \\ 2a_0 &= (-1)^{k+1} Q_{k-2} (Q_{k-2}^2 - (-1)^k) / Q_{k-1} + m Q_{k-1}. \end{aligned}$$

We must, in order to have a_0 positive, require that $m \geq m_0$, where m_0 is the smallest integer for which

$$(-1)^{k+1} Q_{k-2} (Q_{k-2}^2 - (-1)^k) + m_0 Q_{k-1}^2 > 0.$$

This condition on m also suffices to ensure that $N - a_0^2 > 0$ and that $2a_0 \geq N - a_0^2$, allowing a_0 to equal $[\sqrt{N}]$. The only further condition that needs to be met is the requirement that the expression for $2a_0$ has, in fact, even parity so that $a_0 \in \mathbb{Z}$. We shall divide the possibilities into four cases.

Case 1. $Q_{k-2} \equiv 0 \pmod{2}$. In this case $(Q_{k-2}, Q_{k-1}) = 1 \Rightarrow Q_{k-1} \equiv 1 \pmod{2}$ and the even m are the only solutions that will force $2a_0 \equiv 0 \pmod{2}$.

Case 2. $Q_{k-2} \equiv Q_{k-1} \equiv 1 \pmod{2}$. In this case the even m are the only solutions.

Case 3. $Q_{k-2} \equiv 1 \pmod{2}$, $Q_{k-1} \equiv (Q_{k-2}^2 - (-1)^k) / Q_{k-1} \equiv 0 \pmod{2}$. Here all m satisfy the congruence $2a_0 \equiv 0 \pmod{2}$.

Case 4. $Q_{k-2} \equiv (Q_{k-2}^2 - (-1)^k) / Q_{k-1} \equiv 1 \pmod{2}$. This implies that $Q_{k-1} \equiv 0 \pmod{2}$ and there are no solutions $m \in \mathbb{Z}$ which make the formula for $2a_0$ come out even.

Case 4, showing the impossibility of any solutions when both Q_{k-2} and $(Q_{k-2}^2 - (-1)^k) / Q_{k-1}$ are odd, proves the second statement of the theorem. For Cases 1 and 2 we see that N has a continued fraction expansion of the desired form

exactly when

$$\begin{aligned}
N &= \{(-1)^{k+1}Q_{k-2}(Q_{k-2}^2 - (-1)^k)/2Q_{k-1} + bQ_{k-1}\}^2 + 2bQ_{k-2} \\
&\quad + (-1)^{k+1}(Q_{k-2}^2 - (-1)^k)^2/Q_{k-1}^2 \quad \text{for } b \in \mathbf{Z} \text{ and } b \geq b_0 = m_0/2 \\
&= \alpha b^2 + \beta b + \gamma \quad \text{with integral coefficients,} \\
\alpha &= Q_{k-1}^2, \\
\beta &= 2Q_{k-2} - (-1)^k Q_{k-2}(Q_{k-2}^2 - (-1)^k), \\
\gamma &= (Q_{k-2}^2/4 - (-1)^k)(Q_{k-2}^2 - (-1)^k)^2/Q_{k-1}^2.
\end{aligned}$$

Note that the discriminant $\delta = \beta^2 - 4\alpha\gamma = 4(-1)^k$.

For case 3 we see that N has a continued fraction expansion of the desired form exactly when

$$N = \alpha b^2 + \beta b + \gamma \quad \text{for } b \in \mathbf{Z} \text{ and } b \geq b_0 = m_0,$$

where the integral coefficients are

$$\begin{aligned}
\alpha &= Q_{k-1}^2/4, \\
\beta &= Q_{k-2} - (-1)^k Q_{k-2}(Q_{k-2}^2 - (-1)^k)/2, \\
\gamma &= (Q_{k-2}^2/4 - (-1)^k)(Q_{k-2}^2 - (-1)^k)^2/Q_{k-1}^2.
\end{aligned}$$

Here we have $\delta = \beta^2 - 4\alpha\gamma = (-1)^k$. In fact, looking at δ modulo 4 shows that δ must equal 1 and that k must be even for this case to arise.

In summary then we have that all N , and only those, of the form

$$N = N(b) = \alpha b^2 + \beta b + \gamma, \quad b \in \mathbf{Z}, b \geq b_0,$$

are solutions to $\sqrt{N} = ([\sqrt{N}]; a_1, a_2, \dots, a_{k-2} = a_2, a_{k-1} = a_1, a_k = 2[\sqrt{N}])$ for some integers α, β , and γ dependent only on $a_1, a_2, \dots, a_{k-1}$. A notationally different and less direct proof of a similar result may be found in [3, pp. 96–99].

For the special case $k = 1$, we have

$$\sqrt{N} = a_0 + 1/(\sqrt{N} + a_0) \Rightarrow N = a_0^2 + 1$$

allowing us to write, as a solution set,

$$\{N(b) = \alpha b^2 + \beta b + \gamma \mid b \in \mathbf{Z}, b \geq b_0 = 1\},$$

where $\alpha = 1$, $\beta = 0$, and $\gamma = 1$. In this instance, $\delta = \beta^2 - 4\alpha\gamma = -4$. We are now ready for

LEMMA 2. *With notation as above we have:*

For any prime p , $N(n) \equiv 0 \pmod{p^2}$ for at most two n in any residue system modulo p .

PROOF. First consider $p = 2$. As n goes through a complete residue system modulo 4, $N(n)$ takes on the values γ , $\alpha + \beta + \gamma$, $2\beta + \gamma$, and $\alpha - \beta + \gamma$. But if $Q_{k-1} \equiv 1 \pmod{2}$ we have $\alpha \equiv 1 \pmod{2}$ and if $Q_{k-1} \equiv 0 \pmod{2}$ then $\delta = (-1)^k$ implies that $\beta \equiv 1 \pmod{2}$. Using this information one may inspect the four values to verify that at most two of them can be congruent to 0 modulo 4.

Now let p be an odd prime. Assume $p \mid N(b)$, say $N(b) \equiv lp \pmod{p^2}$. Then $N(b + np) \equiv 2abnp + \beta np + lp \pmod{p^2}$. We have $N(b + np) \equiv 0 \pmod{p^2} \Leftrightarrow l + (2\alpha b + \beta)n \equiv 0 \pmod{p}$. But $N(b) \equiv 0 \pmod{p}$ together with $p \nmid \delta$ implies

$p \nmid 2\alpha b + \beta$ and so $(2\alpha b + \beta)$ is invertible modulo p . So $l + (2\alpha b + \beta)n \equiv 0 \pmod{p}$ for exactly one choice of n modulo p . However, $p \mid N(b)$ is possible for at most two choices of $b \pmod{p}$ as $p \nmid \delta \Rightarrow \alpha x^2 + \beta x + \gamma$ is not identically congruent to 0 mod p . Hence there exist at most two n in any residue system modulo p^2 such that $N(n) \equiv 0 \pmod{p^2}$.

We are now ready to conclude the proof of the Theorem with one last lemma which will show that the density of b in $\mathbf{Z}^+$ such that $N(b)$ is squarefree is nonzero.

LEMMA 3. *Define the density $D = \lim_{n \rightarrow \infty} (\text{number of squarefree } N(b) \text{ with } b \leq n)/n$. Then $D > 0$.*

PROOF.

$$\begin{aligned} D &\geq \prod_{p \text{ prime}} \left(1 - \frac{2}{p^2}\right) \geq \prod_{n=2}^{\infty} \left(1 - \frac{2}{n^2}\right) = \exp \left\{ \sum_{n=2}^{\infty} \ln \left(1 - \frac{2}{n^2}\right) \right\} \\ &\geq \exp \left\{ \sum_{n=2}^{\infty} \frac{-4}{n^2} \right\} \quad \text{as } \ln(1 - 2/n^2) > -4/n^2 \text{ for } n \geq 2, \\ &\geq \exp\{-4\pi^2/6\} > 0. \end{aligned}$$

This completes the proof of the lemma and finishes off the proof of the main Theorem. To prove the Corollary it will suffice to exhibit, for each $k \in \mathbf{Z}^+$, a symmetric set of positive integers $\{a_1, \dots, a_{k-1}\}$ such that Q_{k-1} is odd (thus ensuring that one of Q_{k-2} or $(Q_{k-2}^2 - (-1)^k)/Q_{k-1}$ is even).

If $k = 1$ then $N(b) = b^2 + 1$ gives us infinitely many squarefree N . Assume $k > 1$. If $k \not\equiv 0 \pmod{3}$ then set $a_1 = a_{k-1} = 2$ and $a_i = 1$ for $i = 2, \dots, k-2$. If $k \equiv 0 \pmod{3}$ set $a_i = 1$ for $i = 1, \dots, k-1$. In both cases we have the recursion formula for Q_n giving us copies of the Fibonacci sequence ($F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ define the sequence). In the first instance we have $Q_n = F_{n+2}$ for $i = 1, \dots, k-2$ and $Q_{k-1} = 2Q_{k-2} + Q_{k-3} = 2F_k + F_{k-1} + F_{k+2}$. If $k \not\equiv 0 \pmod{3}$ we have $Q_n = F_{n+1}$ for $n = 1, \dots, k-1$, hence $Q_{k-1} = F_k$. But, as F_k is even only when $k \equiv 0 \pmod{3}$, we see that Q_{k-1} is odd in either situation. Therefore, either Q_{k-2} or $(Q_{k-2}^2 - (-1)^k)/Q_{k-1}$ is even and we have satisfied the conditions of the Theorem, thus proving the Corollary.

III. Conclusion. We have shown that, given any symmetric set of positive integers $\{a_1, a_2, \dots, a_{k-2} = a_2, a_{k-1} = a_1\}$ we can find either no integers N (and $a_0 = \lfloor \sqrt{N} \rfloor$) where $\sqrt{N} = (a_0; \overline{a_1, a_2, \dots, a_{k-1}, a_k = 2a_0})$ or we can find infinitely many, distinguishing between the cases by inspecting Q_{k-1} and Q_{k-2} . When there are solutions N they correspond exactly to the positive values taken by a quadratic (with domain $\mathbf{Z}$) whose integer coefficients turn out to be functions only of Q_{k-1} and Q_{k-2} . Among these values there must be infinitely many squarefree integers.

By setting $a_i = 1$ for $i = 1, \dots, k-1$ if $k \not\equiv 0 \pmod{3}$ [and $a_1 = a_{k-1} = 2$, $a_i = 1$ for $i = 2, \dots, k-2$ if $k \equiv 0 \pmod{3}$] it was shown that the conditions of the Theorem are met. Hence, for any $k \in \mathbf{Z}^+$, there are infinitely many squarefree integers N with period equal to k .

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