THE SPACES $H^p(B_n)$, $0 < p < 1$, AND $B_{pq}(B_n)$, $0 < p < q < 1$, ARE NOT LOCALLY CONVEX

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ABSTRACT. In this note, we use the Ryll-Wojtaszczyk polynomials to prove that the spaces $H^p(B_n)$, $0 < p < 1$, and $B_{pq}(B_n)$, $0 < p < q < 1$, fail to be locally convex.

Let B_n denote the unit ball of complex vector space C^n , ∂B_n its boundary and σ_n the positive rotation-invariant measure on ∂B_n with $\sigma_n(\partial B_n) = 1$. By $H(B_n)$ we denote the class of all functions holomorphic in B_n .

The Hardy space H^p , $0 < p < \infty$, is defined on B_n by

$$H^p(B_n) = \{f: f \in H(B_n), \|f\|_p < \infty\},$$

where

$$(1) \quad \|f\|_p = \sup_{0 < r < 1} M_p(r, f), \quad M_p(r, f) = \left\{ \int_{\partial B_n} |f(r\zeta)|^p d\sigma_n(\zeta) \right\}^{1/p}.$$

The space B_{pq} , $0 < p < q < \infty$, is defined on B_n by

$$B_{pq}(B_n) = \{f: f \in H(B_n), \|f\|_{B_{pq}} < \infty\},$$

where

$$(2) \quad \|f\|_{B_{pq}} = \left\{ \int_0^1 (1-r)^{nq(1/p-1/q)-1} M_q^q(r, f) dr \right\}^{1/q}.$$

For $p \geq 1$, $H^p(B_n)$ is a Banach space with norm (1) and for $q \geq 1$, $B_{pq}(B_n)$ is a Banach space with norm (2). For $0 < p < 1$ and $0 < p < q < 1$, we introduce the metrics

$$d(f, g) = \|f - g\|_p^p \quad \text{for } H^p(B_n)$$

and

$$d(f, g) = \|f - g\|_{B_{pq}}^q \quad \text{for } B_{pq}(B_n).$$

These metrics turn $H^p(B_n)$ and $B_{pq}(B_n)$ respectively into F -spaces (complete, metrizable linear topological spaces).

Livingston first noticed that $H^p(U)$ fails to be locally convex for $0 < p < 1$ and U a unit disc, [2]. In this note we will prove that $H^p(B_n)$, $0 < p < 1$, and $B_{pq}(B_n)$, $0 < p < q < 1$, are not locally convex.

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Let $f \in H(B_n)$ with the homogeneous expansion $f(z) = \sum_{k=0}^{\infty} F_k(z)$ and $\beta > 0$; the β th fractional derivative and fractional integral of f are defined respectively by

$$f^{[\beta]}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + \beta + 1)}{\Gamma(k + 1)} F_k(z)$$

and

$$f_{[\beta]}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + 1)}{\Gamma(k + \beta + 1)} F_k(z).$$

In the above, Γ denotes the Gamma function. It is known that $f^{[\beta]}$ and $f_{[\beta]}$ are holomorphic on B_n [3].

In 1983 Ryll and Wojtaszczyk [5] proved the following theorem:

For every $n \geq 1$, there is a sequence $p_1, p_2, \dots$ with $p_k \in P(k, n)$, such that

$$\|p_k\|_{\infty} = 1, \quad \|p_k\|_2 \geq \sqrt{\pi}/2^n, \quad k = 1, 2, \dots,$$

where $P(k, n)$ is the vector space of homogeneous polynomials of degree k in the complex variables $z_1, \dots, z_n$ regarded as functions on ∂B_n and

$$\|p_k\|_{\infty} = \sup_{\zeta \in \partial B_n} |p_k(\zeta)|, \quad \|p_k\|_2 = \left\{ \int_{\partial B_n} |p_k(\zeta)|^2 d\sigma_n(\zeta) \right\}^{1/2}.$$

$\{p_k\}$, $k = 1, 2, \dots$, are called the Ryll-Wojtaszczyk polynomials, for which we have

LEMMA 1. *Let $\{n_k\}$ be a lacunary sequence of positive integers and $\{p_{n_k}\}$ the Ryll-Wojtaszczyk polynomials. If $f(z) = \sum_{k=1}^{\infty} a_k p_{n_k}(z)$ with $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, then for any p , $0 < p < 1$, there is a constant A independent of f , such that*

$$\left\{ \sum_{k=1}^{\infty} |a_k|^2 \right\}^{1/2} \leq A \|f\|_p.$$

PROOF. Since $|p_{n_k}(\zeta)| \leq 1$, $\zeta \in \partial B_n$, $\|p_{n_k}\|_1 \geq \|p_{n_k}\|_2^2 \geq \pi/4^n$. Proposition 1.6 of [5] gives

$$(3) \quad \left\{ \sum_{k=1}^{\infty} |a_k|^2 \right\}^{1/2} \leq A \|f\|_1.$$

Here, and later, A denotes a finite constant, not necessarily the same at each occurrence, independent of f . It is easy to see

$$(4) \quad \|f\|_2^2 \leq \sum_{k=1}^{\infty} |a_k|^2 \|p_{n_k}\|_2^2 \leq \sum_{k=1}^{\infty} |a_k|^2.$$

Combining (3) and (4) we obtain

$$(5) \quad \|f\|_2 \leq A \|f\|_1.$$

On the other hand, for any p , $0 < p < 1$, Hölder's inequality gives

$$(6) \quad \|f\|_1 \leq \|f\|_p^{p/(2-p)} \|f\|_2^{2(1-p)/(2-p)}.$$

From (5) and (6), we have

$$(7) \quad \|f\|_2 \leq A \|f\|_p.$$

Now (3) and (7) give the desired inequality.

A function f , holomorphic in B_n and continuous on $\overline{B}_n$, belongs to Lipschitz class Λ_α , $0 < \alpha \leq 1$, if it satisfies the Lipschitz condition

$$|f(\zeta) - f(\eta)| \leq A|\zeta - \eta|^\alpha, \quad |\zeta - \eta| \rightarrow 0,$$

$\zeta, \eta \in \partial B_n$, A independent of ζ, η .

LEMMA 2. Let $f(z) = \sum_{k=0}^{\infty} F_k(z)$ be the homogeneous expansion of f on B_n . If $f \in \Lambda_\alpha$, $0 < \alpha \leq 1$, then $\sum_{k=1}^{\infty} k^s \|F_k\|_2^2 < \infty$ for any $s < 2\alpha$.

PROOF. Since $f \in \Lambda_\alpha$, by Theorem 6 of [3],

$$(8) \quad |f^{[1]}(r\zeta)| \leq A(1-r)^{\alpha-1}, \quad 0 < r < 1, \zeta \in \partial B_n,$$

where A is independent of r and ζ . Inequality (8) and

$$\int_{\partial B_n} |f^{[1]}(r\zeta)|^2 d\sigma_n(\zeta) = \sum_{k=0}^{\infty} (k+1)^2 r^{2k} \|F_k\|_2^2$$

give $r^{2N} \sum_{k=1}^N k^2 \|F_k\|_2^2 \leq A(1-r)^{2(\alpha-1)}$. Let $r = 1 - N^{-1}$; we have

$$S_N = \sum_{k=1}^N k^2 \|F_k\|_2^2 \leq AN^{2(1-\alpha)}.$$

Hence

$$\begin{aligned} \sum_{k=1}^N k^s \|F_k\|_2^2 &= \sum_{k=1}^N k^2 \|F_k\|_2^2 k^{s-2} = \sum_{k=1}^{N-1} S_k [k^{s-2} - (k+1)^{s-2}] + S_N N^{s-2} \\ &\leq A \sum_{k=1}^{N-1} [k^{s-2\alpha} - (k+1)^{s-2\alpha}] + O(1) = O(1) \end{aligned}$$

for $s < 2\alpha$. The lemma is proved.

THEOREM 1. $H^p(B_n)$, $0 < p < 1$, is not locally convex.

PROOF. Let

$$M = \left\{ f(z) = \sum_{k=1}^{\infty} a_k p_{n_k}(z) : \sum_{k=1}^{\infty} |a_k|^2 < \infty \right\},$$

where $\{n_k\}$ is a lacunary sequence of natural numbers and $\{p_{n_k}\}$ are the Ryll-Wojtaszczyk polynomials. It is clear that M is a subspace of $H^p(B_n)$, $0 < p < 1$. Take a sequence $\{c_k\}$ such that $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ and for any $\varepsilon > 0$, $\sum_{k=1}^{\infty} k^\varepsilon |c_k|^2 = \infty$. Define a linear functional φ on M as follows:

$$\varphi(f) = \sum_{k=1}^{\infty} a_k c_k, \quad f(z) = \sum_{k=1}^{\infty} a_k p_{n_k}(z) \in M.$$

The Schwarz Lemma and Lemma 1 give

$$|\varphi(f)| \leq A \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq A \|f\|_p, \quad f \in M.$$

This proves that φ is a continuous linear functional on M . If $H^p(B_n)$, $0 < p < 1$, is locally convex, then there exists a continuous linear functional Φ on $H^p(B_n)$, such that $\Phi(f) = \varphi(f)$ on M [4, Theorem 3.6]. By Theorem 9 of [3], there is some $g \in \Lambda_\alpha$, $0 < \alpha \leq 1$, such that

$$(9) \quad \Phi(f) = \lim_{r \rightarrow 1} \int_{\partial B_n} f(r\zeta) \overline{g(\zeta)} d\sigma_n(\zeta)$$

for all $f \in H^p(B_n)$. Let $g(z) = \sum_{k=0}^{\infty} F_k(z)$; Lemma 2 gives

$$(10) \quad \sum_{k=1}^{\infty} k^s \|F_k\|_2^2 < \infty, \quad s < 2\alpha.$$

Take $f = p_{n_k}$ in (9); since $\Phi(f) = \varphi(f) = c_k$, we have $|c_k| \leq \|F_{n_k}\|_2$, $k = 1, 2, \dots$. Now (10) gives

$$\sum_{k=1}^{\infty} k^s |c_k|^2 \leq \sum_{k=1}^{\infty} n_k^s \|F_{n_k}\|_2^2 < \infty, \quad s < 2\alpha.$$

This contradiction proves that $H^p(B_n)$ fails to be locally convex.

THEOREM 2. $B_{pq}(B_n)$, $0 < p < q < 1$, is not locally convex.

Before proving the theorem, we first prove the following lemma.

LEMMA 3. Suppose $0 < p < q < 1$ and $\beta = n(1/p - 1/q)$. If $f^{[\beta]} \in B_{pq}(B_n)$, then $f \in H^q(B_n)$ and there exists a constant A independent of f , such that $\|f\|_q \leq A\|f^{[\beta]}\|_{B_{pq}}$.

PROOF. A direct computation gives

$$f(r\zeta) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-\rho)^{\beta-1} f^{[\beta]}(r\rho\zeta) d\rho$$

for $0 < r < 1$ and $\beta > 0$, and

$$|f(r\zeta)| \leq \frac{1}{\Gamma(\beta)} \sum_{k=1}^{\infty} \int_{\rho_{k-1}}^{\rho_k} (1-\rho)^{\beta-1} |f^{[\beta]}(r\rho\zeta)| d\rho,$$

where $\rho_k = 1 - 2^{-k}$. Let $H_k(r\zeta) = \sup\{|f^{[\beta]}(r\rho\zeta)| : \rho_{k-1} < \rho < \rho_k\}$. Then

$$(11) \quad |f(r\zeta)|^q \leq A \sum_{k=1}^{\infty} H_k^q(r\zeta) 2^{-q\beta k}.$$

Here we have used the condition $0 < q < 1$. Now Theorem 3 of [1] and the monotonicity of the mean give

$$\begin{aligned} M_q^q(r, f) &\leq A \sum_{k=1}^{\infty} 2^{-q\beta k} \sup\{M_q^q(r\rho, f^{[\beta]}): \rho_{k-1} < \rho < \rho_k\} \\ &\leq A \sum_{k=1}^{\infty} 2^{-q\beta k} M_q^q(r\rho_k, f^{[\beta]}) \leq A \int_0^1 (1-\rho)^{\beta q-1} M_q^q(\rho, f^{[\beta]}) d\rho. \end{aligned}$$

Taking $\beta = n(1/p - 1/q)$ in the above inequality, the desired result follows.

PROOF OF THEOREM 2. Let $\beta = n(1/p - 1/q)$ and $E = \{h: h \in H^p(B_n), h_{[\beta]} \in M\}$. Taking $k = q$ in Theorem 4 of [3], we have

$$(12) \quad \|f\|_{B_{pq}} \leq A\|f\|_p, \quad 0 < p < q < \infty.$$

This shows that E is a subspace of $B_{pq}(B_n)$. It is clear that for every $h \in E$, there exists an $f \in M$ with $f^{[\beta]} = h$, hence h can be written as

$$(13) \quad h(z) = \sum_{k=1}^{\infty} \frac{\Gamma(n_k + \beta + 1)}{\Gamma(n_k + 1)} a_k p_{n_k}(z).$$

Take a sequence $\{b_k\}$ such that $\sum_{k=1}^{\infty} n_k^{2\beta} |b_k|^2 < \infty$, and $\sum_{k=1}^{\infty} n_k^{2\beta+\varepsilon} |b_k|^2 = \infty$ for any $\varepsilon > 0$. Define a linear functional on E as follows:

$$(14) \quad \psi(h) = \sum_{k=1}^{\infty} \frac{\Gamma(n_k + \beta + 1)}{\Gamma(n_k + 1)} a_k b_k, \quad h \in E.$$

The series in (14) is convergent since $\Gamma(n_k + \beta + 1)/\Gamma(n_k + 1) \approx n_k^\beta$ and

$$(15) \quad |\psi(h)| \leq A \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}, \quad h \in E.$$

Since $f^{[\beta]} = h \in H^p(B_n) \subset B_{pq}(B_n)$, according to Lemma 3, there is a constant A such that

$$(16) \quad \|f\|_q \leq A\|f^{[\beta]}\|_{B_{pq}} = A\|h\|_{B_{pq}}.$$

(15), (16) and Lemma 1 give $|\psi(h)| \leq A\|h\|_{B_{pq}}$ for $h \in E$. This proves that ψ is a continuous linear functional on the subspace E of $B_{pq}(B_n)$. If $B_{pq}(B_n)$ is locally convex, then there is a continuous linear functional Ψ on $B_{pq}(B_n)$ with $\Psi(h) = \psi(h)$ on E . By (12), Ψ is also a continuous linear functional on $H^p(B_n)$. Suppose p satisfies $n/(n+l) \leq p < n/(n+l-1)$, l a natural number. By Theorem 9 of [3], there exists a v with $v^{[l-1]} \in \Lambda_\alpha$, $\alpha = n(1/p - 1) - l + 1$, such that

$$(17) \quad \Psi(h) = \lim_{r \rightarrow 1} \int_{\partial B_n} h(r\zeta) \overline{v(\zeta)} d\sigma_n(\zeta)$$

for every $h \in H^p(B_n)$. Let $v(z) = \sum_{k=0}^{\infty} G_k(z)$; Lemma 2 tells us

$$(18) \quad \sum_{k=1}^{\infty} k^{s+2(l-1)} \|G_k\|_2^2 < \infty, \quad s < 2\alpha.$$

Take $h = (\Gamma(n_k + \beta + 1)/\Gamma(n_k + 1))p_{n_k}$ in (17); since

$$\Psi(h) = \psi(h) = (\Gamma(n_k + \beta + 1)/\Gamma(n_k + 1))b_k,$$

we have

$$(19) \quad |b_k| \leq \|G_{n_k}\|_2, \quad k = 1, 2, \dots$$

Choose $\eta > 0$ such that $\eta + 2\beta < 2n(1/p - 1) - \eta$ and let $s = 2\alpha - \eta$. Then $s + 2(l-1) = 2n(1/p - 1) - \eta > \eta + 2\beta$. Now (18) and (19) give

$$\sum_{k=1}^{\infty} n_k^{\eta+2\beta} |b_k|^2 \leq \sum_{k=1}^{\infty} n_k^{s+2(l-1)} \|G_{n_k}\|_2^2 < \infty.$$

This is a contradiction. The theorem is proved.

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