STRONG UNICITY OF BEST APPROXIMATIONS IN $L_\infty(S, \Sigma, \mu)$

RYSZARD SMARZEWSKI

(Communicated by Richard R. Goldberg)

ABSTRACT. We prove that the set of functions in a Banach space $L_\infty(S, \Sigma, \mu)$, which have a strongly unique best approximation in a finite dimensional subspace, is dense in the set of functions having a unique best approximation in the subspace.

1. Introduction. Let M be a finite dimensional subspace of a real Banach space X . An element $m \in M$ is said to be a unique best approximation in M to an element $x \in X$ if

$$(1.1) \quad \|x - m\| < \|x - y\|$$

for all $y \neq m$ in M . Such an element m is called a strongly unique best approximation in M to x if there exists a constant $c > 0$ such that

$$(1.2) \quad \|x - m\| \leq \|x - y\| - c\|m - y\|$$

for all y in M . Denote the set of all elements $x \in X$ having a (strongly) unique best approximation in M by U_M (SU_M , respectively). Clearly, we have $U_M \supset SU_M \supset M$.

It is well known that $U_M \neq SU_M$ in general (see [3, p. 82]). On the other hand, Nürnberger and Singer [5] showed that SU_M is dense in U_M for any finite dimensional subspace M of the space $C_0(T)$, where T is a locally compact metric space. Recently, the same result has been proved for the space $L_1(S, \Sigma, \mu)$ by Angelos and Schmidt [1, 2]. In §2 of this note we first prove the former result provided only that T is a locally compact Hausdorff space. This is used in §3 to show the density of SU_M in U_M for any finite dimensional subspace M of the space $L_\infty(S, \Sigma, \mu)$.

In order to simplify the proofs we note that the best constant $c = c_M(x)$ in (1.2) is equal to

$$(1.3) \quad c_M(x) = \inf \left\{ \frac{\|x - y\| - \|x - m\|}{\|m - y\|} : y \in M, y \neq m \right\}.$$

This, in conjunction with the well-known inequality [4, p. 446]

$$\frac{\|\hat{x} + \alpha y\| - \|\hat{x}\|}{\|\alpha y\|} \leq \frac{\|\hat{x} + y\| - \|\hat{x}\|}{\|y\|}, \quad 0 < \alpha \leq 1,$$

implies that

$$(1.4) \quad c_M(x) = c_M(\hat{x}) = \inf \left\{ \frac{\|\hat{x} + y\| - \|\hat{x}\|}{\|y\|} : y \in M, 0 < \|y\| \leq \varepsilon \right\}$$

Received by the editors February 20, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 41A45, 41A52; Secondary 46E15, 46E30.

for every $\varepsilon > 0$, where the element $\hat{x} = x - m$ has 0 as a unique best approximation in M if and only if m is a unique best approximation in M to x .

2. Strong unicity in $C_0(T)$. Let $C_0(T)$ be the Banach space of all real-valued continuous functions x on a locally compact Hausdorff space T such that the set $\{t \in T: |x(t)| \geq \varepsilon\}$ is compact for each $\varepsilon > 0$ with the norm $\|x\| = \sup\{|x(t)|: t \in T\}$. In particular, if T is a compact Hausdorff space, we shall also write $C(T)$ instead of $C_0(T)$. For a function $x \in C_0(T) \setminus \{0\}$ and a positive integer n define nonempty compact subsets T_n of T and continuous functionals f_n by

$$(2.1) \quad T_n = \{t \in T: |x(t)| \geq (1 - 1/(n+1))\|x\|\}$$

and

$$(2.2) \quad f_n(y) = \max_{t \in T_n} [y(t) \operatorname{sgn} x(t)], \quad y \in C_0(T).$$

LEMMA 2.1. *If 0 is a unique best approximation in a finite dimensional subspace M of $C_0(T)$ to a function $x \neq 0$ in $C_0(T)$, then there exists a constant $a_n > 0$ such that*

$$f_n(y) \geq a_n \|y\| \quad \text{for all } y \text{ in } M.$$

PROOF. The inequality is trivial in the case when $y = 0$. Otherwise, by the continuity of f_n and compactness of the unit sphere in M , it is sufficient to show the inequality $f_n(y) > 0$ for all $y \neq 0$ in M . Suppose that this inequality is not true for an element $y \neq 0$ and choose a real number $\alpha = \alpha(n) > 0$ such that $\alpha\|y\| \leq \|x\|/(n+1)$. Then we have $|\alpha y(t)| \leq |x(t)|$ and

$$|x(t) + \alpha y(t)| = |x(t)| + \alpha y(t) \operatorname{sgn} x(t) \leq |x(t)| \leq \|x\|$$

for all $t \in T_n$. On the other hand, if $t \in T \setminus T_n$, then

$$|x(t) + \alpha y(t)| \leq |x(t)| + |\alpha y(t)| \leq \left(1 - \frac{1}{n+1}\right) \|x\| + \frac{\|x\|}{n+1} = \|x\|.$$

Thus $-\alpha y \neq 0$ is a best approximation in M to x for sufficiently small $\alpha > 0$ which contradicts the fact that 0 is a unique best approximation in M to x . Q.E.D.

This lemma will be applied to prove the following theorem which generalizes Theorem 3.5 [5].

THEOREM 2.1. *Let m be a unique best approximation in a finite dimensional subspace M of $C_0(T)$ to a function x in $C_0(T)$. Then there exist a sequence $\{x_n\}$ in $C_0(T)$, convergent uniformly to x and constants $c_n > 0$ such that*

$$\|x_n - m\| \leq \|x_n - y\| - c_n \|m - y\|$$

for all y in M .

PROOF. If $x = m$, then we can put $x_n = m$ and $c_n = 1$. Otherwise, by virtue of (1.4), we may suppose without loss of generality that 0 is a unique best approximation in M to $x \neq 0$ in $C_0(T)$. By the Tietze-Urysohn theorem [6, p. 101], there exist real-valued continuous functions x_n defined on T_n and such that

$$x_n(t) = \begin{cases} \|x\| \operatorname{sgn} x(t) & \text{if } t \in T_{n+2}, \\ x(t) & \text{if } t \in T_n \cap (\overline{T \setminus T_{n+1}}), \end{cases}$$

and

$$(1 - 1/(n+1))\|x\| \leq x_n(t) \leq \|x\|, \quad t \in T_n,$$

where the nonempty compact sets $T_1 \supset T_2 \supset \dots$ are defined as in (2.1). We extend these functions to functions x_n in $C_0(T)$ by

$$x_n(t) = x(t), \quad t \in T \setminus T_n.$$

Hence, we get

$$|x(t) - x_n(t)| \leq \|x\| - x_n(t) \leq \|x\|/(n+1)$$

for all $t \in T$. Thus, x_n converges uniformly to x as $n \rightarrow \infty$. Now suppose that $y \in M$ and $0 < \|y\| \leq \|x\|/(n+1)$. Then we have

$$|y(t)| \leq (1 - 1/(n+1))\|x\| \leq |x(t)| \leq |x_n(t)|$$

and

$$\|x_n + y\| \geq |x_n(t) + y(t)| = |x_n(t)| + y(t)\operatorname{sgn} x_n(t) = \|x_n\| + y(t)\operatorname{sgn} x(t)$$

for all $t \in T_{n+2}$. This in conjunction with (2.2) and Lemma 2.1 implies that

$$\|x_n + y\| \geq \|x_n\| + f_{n+2}(y) \geq \|x_n\| + a_{n+2}\|y\|,$$

where the constant $a_{n+2} > 0$ is independent of y . Hence we can apply (1.4) in order to finish the proof. Q.E.D.

From Theorem 2.1 we immediately obtain

COROLLARY 2.1. *If M is a finite dimensional subspace of the space $C_0(T)$, then SU_M is dense in U_M .*

Since $SU_M \subset U_M$, it follows from Corollary 2.1 that $\overline{SU_M} = \overline{U_M}$. However, in the particular case when T is a discrete set, i.e., $C_0(T) = c_0(T)$, we have $T_n = T_{n+2} = \{t \in T: |x(t)| = \|x\|\}$ and $x_n(t) \equiv x(t)$ for all sufficiently large n . Therefore, Theorem 2.1 implies

COROLLARY 2.2. *If M is a finite dimensional subspace of the space $c_0(T)$, then $SU_M = U_M$.*

3. Strong unicity in $L_\infty(S, \Sigma, \mu)$. For a positive measure space (S, Σ, μ) , let $L_\infty(S, \Sigma, \mu)$ denote the Banach space of all μ -essentially bounded real-valued functions x on S with the norm

$$\|x\| = \mu\text{-ess sup}_{s \in S} |x(s)|.$$

THEOREM 3.1. *If M is a finite dimensional subspace of the space $L_\infty(S, \Sigma, \mu)$, then SU_M is dense in U_M .*

PROOF. By Theorem 11 [4, p. 445], there exist a compact Hausdorff space T and an isometric isomorphism J between $L_\infty(S, \Sigma, \mu)$ and $C(T)$. Using (1.1) and (1.3) this then yields $U_M = U_G$ and $c_M(x) = c_G(Jx)$ for any element x and any finite dimensional subspace M of $L_\infty(S, \Sigma, \mu)$, where $G = J(M)$ is a finite dimensional subspace of $C(T)$. Hence $SU_M = SU_G$, and so the theorem follows readily from Corollary 2.1. Q.E.D.

REFERENCES

1. J. Angelos and D. Schmidt, *Strong uniqueness in $L^1(X, \Sigma, \mu)$* , Approximation Theory. IV, Academic Press, New York, 1983.
2. ———, *The prevalence of strong uniqueness in L^1* , preprint.
3. E. W. Cheney, *Introduction to approximation theory*, McGraw-Hill, New York, 1966.
4. N. Dunford and J. T. Schwartz, *Linear operators*. I, Interscience, New York, 1958.
5. G. Nürnberger and I. Singer, *Uniqueness and strong uniqueness of best approximations by spline subspaces and other subspaces*, J. Math. Anal. Appl. **90** (1982), 171–184.
6. Z. Semadeni, *Banach spaces of continuous functions*, PWN, Warsaw, 1971.

DEPARTMENT OF MATHEMATICS, M. CURIE-SKŁODOWSKA UNIVERSITY, 20-031
LUBLIN, POLAND