ON SEMIREGULAR POINTS OF THE MARTIN BOUNDARY

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(Communicated by Irwin Kra)

ABSTRACT. For a selfadjoint, Brelot harmonic space with Martin boundary, call "semiregular" a point z of the boundary at which the Green function has a positive upper limit; call "maximal" the filter $\mathcal{F}_z$ converging to z along which that upper limit is attained. If such a z is minimal then any bounded harmonic function has a limit along $\mathcal{F}_z$ and $\mathcal{F}_z$ is coarser than the fine filter associated with z . The semiregular points form a set of harmonic measure zero.

In the context of a selfadjoint, Brelot harmonic space for which a Martin boundary is defined, we call *semiregular* those points of the boundary at which the "Green function" (and thus any positive potential with compact carrier) has a positive upper limit. If a semiregular point z is minimal, then along the coarsest filter $\mathcal{F}_z$ for which the above upper limit is attained, any bounded harmonic function has a limit. We further show that $\mathcal{F}_z$ is coarser than the fine filter associated with z ; this result generalizes for bounded harmonic functions a result of Naïm [12]. We also prove that for any minimal point z of the Martin boundary and any potential q with compact carrier, q divided by the Martin kernel for z has a limit 0 at z . Finally we prove that the semiregular points form a set of harmonic measure zero. We begin in a more general context.

Let X be a locally compact but not compact, connected and locally connected Hausdorff space on which is defined a harmonic sheaf in the sense of Brelot. We assume that 1 is harmonic on X . Let P_0 denote the set of nonzero potentials on X with compact carrier. Recall that from the domination principle it follows that for any p and q in P_0 , the ratio p/q is bounded and bounded away from zero in the complement of a compact subset of X . Let $\bar{X}$ be the compactification of X such that for any two continuous elements p and q in P_0 , the ratio p/q has a real-valued, continuous extension to $\bar{X}$ with the set of such extensions separating points of $\bar{X} - X$. The extension is also denoted by p/q . It is easy to show that $\bar{X}$ is the Martin compactification of X when X is a domain in Euclidean space with the usual harmonic functions as the harmonic class. If, moreover, X is a Lipschitz domain then by a result of Hunt and Wheeden [9], $\bar{X}$ is homeomorphic to the topological closure of X . We assume in general that our harmonic space has sufficient structure so that $\bar{X}$ is a resolutive compactification of X . (See [7].)

We denote the boundary $\bar{X} - X$ by Δ . For any point $z \in \bar{X}$ and subset A of $\bar{X}$, we write $\lim_A f(z)$ for the limit of a function f with respect to the trace on

Received by the editors February 24, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 31D05, 31C35.

Key words and phrases. Martin boundary, Fine neighborhood, harmonic boundary.

The work of both authors was supported in part by a grant from the U.S. National Science Foundation DMS 84-01294.

A of the neighborhood filter of z ; similar notation is used for $\limsup$ and $\liminf$. We will write $H(f)$ for the Dirichlet solution on X of a bounded Borel measurable f on Δ , and we write μ_x for the harmonic measure on Δ of a point $x \in X$; i.e., $\mu_x(\Delta) = 1$ and $H(f)(x) = \int f d\mu_x$.

Fix a pair p, q in P_0 . Since $p = (p/q)q$, if $\mathcal{F}$ is a filter in X converging to $z \in \Delta$ with $\lim_{\mathcal{F}} p = \limsup_X p(z)$, then $\lim_{\mathcal{F}} q = \limsup_X q(z)$. Moreover, if $\limsup_X p(z) = 0$ then $\limsup_X q(z) = 0$. Thus the following definition is independent of the choice of $p \in P_0$.

DEFINITION. A point $z \in \Delta$ is called *semiregular* if for some $p \in P_0$,

$$\limsup_X p(z) > 0.$$

Fix a semiregular $z \in \Delta$ and a $p \in P_0$. The filter on X generated by the sets $\{p > \alpha\} \cap U$, where α is a positive number smaller than $\limsup_X p(z)$ and U is an open neighborhood of z , is called the *maximal filter* at z .

Let p be a continuous element of P_0 . For any positive superharmonic function s on X , we denote by s/p the pointwise supremum on $\bar{X}$ of the directed family of continuous functions $\{q/p : q \in P_0, q \text{ continuous and } q \leq s \text{ on } X\}$. We extend the ratio $0/p$ with 0. Clearly s/p is lower semicontinuous on $\bar{X}$, and (using the Riesz decomposition property) for any positive $\alpha \in \mathbf{R}$ and any nonnegative superharmonic t on X , $(\alpha s + t)/p = \alpha(s/p) + t/p$ on $\bar{X}$.

DEFINITION. Given a continuous $p \in P_0$, we let M_p denote the set of all continuous functions on Δ which are restrictions to Δ of functions having the form q/p where $q \in P_0$. We also denote by $D(M_p)$ the linear space $M_p - M_p$ formed from the convex cone M_p .

PROPOSITION 1. Fix a continuous $p \in P_0$. The convex cone M_p satisfies the strong Riesz interpolation property, i.e., given f_1, f_2, g_1, g_2 in M_p with $f_1 \vee f_2 < g_1 \wedge g_2$, there is an $f \in M_p$ such that $f_1 \vee f_2 < f < g_1 \wedge g_2$.

PROOF. Fix s_i and t_i in P_0 , $i = 1, 2$, so that s_i/p extends to f_i on Δ and t_i/p extends to g_i . Choose θ so that $f_1 \vee f_2 < \theta \cdot (g_1 \wedge g_2)$ on Δ . Obviously the set where $s_1/p \vee s_2/p \geq \theta \cdot (t_1/p \wedge t_2/p)$ is a compact subset K of X . Let $t = t_1 \wedge t_2$; K is the set where $s_1 \vee s_2 \geq \theta \cdot t$. Thus $s_i \wedge \theta \cdot t$ is also in P_0 and equals s_i outside of K for $i = 1, 2$. We may therefore assume that $s_i \leq \theta \cdot t$ for $i = 1, 2$. Therefore on Δ

$$f_1 \vee f_2 = (s_1/p) \vee (s_2/p) \leq \theta \cdot t/p < t/p \leq g_1 \wedge g_2.$$

Let Q denote the family of continuous functions $q \in P_0$ with $q < t$. Since Q is an increasing family, there is a $q \in Q$ with $f_1 \vee f_2 < q/p$ on Δ . Since $q < t$ on X and q has compact support, for some $\varepsilon > 0$, $q + \varepsilon p < t$ on X . Thus $q/p < t/p$ on $\bar{X}$. The restriction to Δ of q/p is the desired interpolating function. $\square$

Given the interpolation property for M_p , it follows easily that M_p also has a strong Riesz decomposition property; i.e., given f, g_1, g_2 in M_p with $f < g_1 + g_2$ there exist functions f_1 and f_2 in M_p with $f = f_1 + f_2$ and $f_i \leq g_i$ for $i = 1, 2$.

Recall that the Choquet boundary of Δ with respect to $D(M_p)$ is the set of all points $z \in \Delta$ for which unit mass at z is the only Borel measure on Δ that evaluates each function in $D(M_p)$ at z . Since for a continuous $q \in P_0$, $D(M_q) = (p/q) \cdot D(M_p)$, the Choquet boundary of Δ with respect to $D(M_p)$ is independent of the choice of p . We denote this Choquet boundary by Δ_c . If s is a positive superharmonic

function on X , then by the lower semicontinuity of the function s/p on $\overline{X}$ we have at each $z \in \Delta$, $(s/p)(z) \leq \liminf_X (s/p)(z)$. We show now that we actually have equality at points of the Choquet boundary; this will be quite important in the results that follow.

PROPOSITION 2. *Fix a continuous $p \in P_0$ and let z be a point in the Choquet boundary Δ_c . For any positive superharmonic function s we have $(s/p)(z) = \liminf_X (s/p)(z)$.*

PROOF. Denote by P_0^c the set of continuous functions in P_0 , and let F be the linear space of functions f on $\overline{X}$ of the form $f = (p_1/p) - (p_2/p)$, where p_1 and p_2 are functions in P_0^c . The Choquet boundary of $\overline{X}$ with respect to F coincides with $X \cup \Delta_c$. By applying first a result of H. Bauer [2] and then the theory of reduced functions, we have for any p_1 and p_2 in P_0^c ,

$$\begin{aligned} (p_1/p) \wedge (p_2/p) &= \sup\{f \in F: f \leq (p_1/p) \wedge (p_2/p)\} \\ &= \sup\{q/p: q \in P_0^c, q \leq p_1 \wedge p_2\} \\ &= (p_1 \wedge p_2)/p \end{aligned}$$

on the set $X \cup \Delta_c$. Given a positive superharmonic s with $\alpha = \liminf_X (s/p)(z)$, we have $0 \leq (s/p)(z) \leq \alpha$. We may assume that $\alpha > 0$. Fix β with $0 < \beta < \alpha$, and choose a neighborhood U of z such that $s/p > \beta$ on $U \cap X$. Functions of the form q/p , $q \in P_0^c$, generate the topology of $\overline{X}$. Therefore there is a continuous function φ on $\overline{X}$ such that $0 \leq \varphi \leq 1$, $\varphi(z) = 1$, φ vanishes off of U and φ is the difference $f - g$ where $f = (p_1/p) \wedge (p_2/p) \wedge \cdots \wedge (p_k/p)$ and $g = (q_1/p) \wedge (q_2/p) \wedge \cdots \wedge (q_l/p)$, each $p_i, q_j \in P_0^c$ (see [4, p. 70, Lemma 3.1.2]). Thus on X we have $\beta(f - g) \leq s/p$. Writing $u = p_1 \wedge p_2 \wedge \cdots \wedge p_k$ and $v = q_1 \wedge q_2 \wedge \cdots \wedge q_l$, we have on X , and therefore on $\overline{X}$, the inequality $\beta(u/p - v/p) \leq s/p$. Using the first part of the proof, we have $f(z) = (u/p)(z)$, $g(z) = (v/p)(z)$, whence

$$\beta = \beta(f(z) - g(z)) = \beta((u/p)(z) - (v/p)(z)) \leq (s/p)(z).$$

The assertion follows since β is arbitrary. $\square$

THEOREM 1. *Fix a semiregular point $z \in \Delta_c$. For any bounded harmonic function h on X , we have $\lim_{\mathcal{F}} h$ exists where $\mathcal{F}$ is the maximal filter at z . It follows that the harmonic measures for points of X have a weak* limit μ_z along the maximal filter at z . If the measure μ_z is unit mass at z , then the set of solutions $\{H(f): f \in C(\Delta)\}$ is one dimensional.*

PROOF. Fix a continuous $p \in P_0$ with $\limsup_X p(z) = 1$. Let h be a harmonic function with $0 \leq h \leq 1$. It follows from the definition of $\mathcal{F}$ that p is continuous at z along $\mathcal{F}$. It therefore suffices to show that h/p is continuous at z along $\mathcal{F}$, and for that we use Proposition 2:

$$\begin{aligned} (1/p)(z) &= 1 = \lim_{\mathcal{F}} 1/p, \\ (h/p)(z) &= \liminf_X (h/p)(z) \leq \liminf_{\mathcal{F}} (h/p), \\ 1 - (h/p)(z) &= ((1 - h)/p)(z) \\ &\leq \liminf_{\mathcal{F}} ((1/p) - (h/p)) = 1 - \limsup_{\mathcal{F}} h/p. \end{aligned}$$

It now follows that for any continuous f on Δ , the limit of $\int f d\mu_x$ exists along $\mathcal{F}$, i.e., μ_x has a weak* limit μ_z . For the sake of contradiction, assume μ_z equals unit mass δ_z at z and there is a nonconstant bounded harmonic function on X of the form $H(f)$ for $f \in C(\Delta)$. We may assume that f is a nonnegative function with $f(z) = 0$ and $H(f) \neq 0$. By the domination principle, there is a positive α with $p \leq \alpha H(f)$, whence $\lim_{\mathcal{F}} p \leq \alpha \cdot \lim_{\mathcal{F}} H(f) = 0$, contradicting the definition of $\mathcal{F}$. $\square$

We will assume for the rest of the paper that there exists a kernel function $G(x, y)$ on $X \times X$ such that

- (i) G is lower semicontinuous on the product $X \times X$;
- (ii) for any $x \in X$, $G(x, \cdot)$ is a potential on X harmonic on $X - \{x\}$;
- (iii) for any $p \in P_0$, there is a measure ρ such that

$$p(y) = \int G(x, y) d\rho(x);$$

- (iv) for all $x, y \in X$, $G(x, y) = G(y, x)$.

It follows from assumptions (ii) and (iii) that potentials with single point carrier are proportional. Given assumption (iv), the results to follow apply only to selfadjoint harmonic spaces.

We now fix for the rest of this paper a point $x_0 \in X$ and a continuous potential $p \in P_0$ such that on the complement of some compact subset of X , $p = G(x_0, \cdot)$. We will denote by K the mapping on $X \times \bar{X}$ defined by $K(x, y) = (G(x, \cdot)/p)(y)$. For any $y \in \Delta$, the function $K(\cdot, y)$ is a positive harmonic function on X taking the value 1 at x_0 . It is easy to see that $\bar{X}$ can be identified with the Martin compactification of X associated with the kernel K [7]. It follows from results in [8] that the Martin boundary Δ is metrizable and each point of Δ has a countable neighborhood system in $\bar{X}$. We denote by Δ_1 the set of points $y \in \Delta$ for which $K(\cdot, y)$ is minimal. We further denote by L the convex cone of functions on Δ of the form $\sum_1^n \alpha_i K(x_i, \cdot)$; here $\alpha_i \geq 0$ in $\mathbf{R}$ and $x_i \in X$, $1 \leq i \leq n$. Note that $L \subset M_p$. We will denote by $D(L)$ the linear space $L - L$.

PROPOSITION 3. *Let $\bar{L}$ denote the uniform closure of L . We then have $L \subset M_p \subset \bar{L}$. Consequently the Choquet boundary Δ_c with respect to $D(M_p)$ is also the Choquet boundary with respect to $D(L)$. Moreover, the convex cones L and $\bar{L}$ satisfy the Riesz interpolation property.*

PROOF. The first assertion follows by using the equicontinuity properties of harmonic functions, a finite Riemann sum approximation of the integral with respect to the measure ρ given by property (iii) for the kernel G , and the domination principle. The third assertion then follows from Proposition 1. $\square$

PROPOSITION 4. *The Choquet boundary Δ_c coincides with the set Δ_1 of minimal points in Δ .*

PROOF. Given $z \in \Delta$ with δ_z denoting unit mass at z , the proposition follows from the equivalence of the following four statements:

- (i) z is a minimal point, i.e., $K(\cdot, z)$ is a minimal function.
- (ii) If μ is a Borel measure on Δ such that $\int K(x, \cdot) d\mu = K(x, z)$ for all $x \in X$, then $\mu = \delta_z$.

(iii) If μ is a Borel measure on Δ such that $\int f d\mu = f(z)$ for all $f \in D(L)$ then $\mu = \delta_z$.

(iv) z is a point of the Choquet boundary of $D(L)$. $\square$

COROLLARY. *Theorem 1 holds for semiregular points in Δ_1 .*

Recall that for a point $z \in \Delta_1$, the filter of fine neighborhoods at z is the collection of all subsets $U \subset X$ such that the balayage of $K(\cdot, z)$ over $X - U$ is different from $K(\cdot, z)$. The trace on X of any ordinary neighborhood of z in $\overline{X}$ is a fine neighborhood at z . The next proposition taken together with Theorem 1 extends for bounded harmonic functions a result stated by L. Naïm [12] for fine filters.

THEOREM 2. *Let z be a minimal, semiregular point of the Martin boundary Δ . The maximal filter at z is coarser than the filter of fine neighborhoods at z , and the corresponding function $K(\cdot, z)$ is unbounded.*

PROOF. Fix $\varepsilon > 0$ with $\varepsilon < \alpha = \limsup_X G(x_0, \cdot)(z)$, and let $U_\varepsilon = \{x \in X : G(x_0, x) > \varepsilon\}$. Since by property (iii) for the kernel G , $G(x_0, \cdot)$ lies on an extreme ray in the set of positive potentials, U_ε is a connected set. We set $U = U_\varepsilon$ if $x_0 \in U_\varepsilon$; otherwise we form U by adjoining a sufficiently large, connected, relatively compact open subset of X to U_ε so that the union is connected and contains x_0 . Denote by μ_x^U harmonic measure for U with respect to a point $x \in U$. Let h and G' be the functions defined on $U \times U$ by $h(x, y) = \int G(x, \cdot) d\mu_y^U$, and $G' = G - h$. It is easy to see that the function h is harmonic in each variable. On that part of the boundary of U contained in Δ , the potential $G(x_0, \cdot)$ has a $\liminf$ at least ε , and so by the usual construction of an Evans potential using an exhaustion of X , there is another potential on X with infinite $\liminf$ at the same set of boundary points. On that part of the boundary of U contained in X , $G(x_0, \cdot) \leq \varepsilon$. It follows that $h(x_0, \cdot)$ is dominated by the constant function ε on U . Let y_n , $n \in \mathbb{N}$, be a sequence in U converging to z with $G(x_0, y_n)$ converging to α . Then $\liminf_n G'(x_0, y_n) \geq \alpha - \varepsilon > 0$. Since $K(\cdot, y_n) \cdot p(y_n) = G(\cdot, y_n)$, $n \in \mathbb{N}$, the sequence $G(\cdot, y_n)$ converges pointwise on U . By taking a subsequence if necessary, we may assume that the sequence $h(\cdot, y_n)$ and, therefore, the sequence $G'(\cdot, y_n)$ converge pointwise to harmonic functions u and v respectively on U . Since $v(x_0) \geq \alpha - \varepsilon$, v is strictly positive on U . Now we have

$$K(\cdot, z) \cdot \alpha = \lim_n K(\cdot, y_n) \cdot p(y_n) = \lim_n [G'(\cdot, y_n) + h(\cdot, y_n)] = v + u.$$

On U the balayage of $K(\cdot, z)$ over $X - U$ equals the Dirichlet solution in U with respect to the boundary function $K(\cdot, z)$, i.e., this is the function w on U which at each $x \in U$ has the value $w(x) = \int K(\cdot, z) d\mu_x^U$. By Fatou's Lemma,

$$\begin{aligned} w(x) &= \int \lim_n K(\cdot, y_n) d\mu_x^U \leq \liminf_n \int K(\cdot, y_n) d\mu_x^U \\ &= \lim_n h(x, y_n)/p(y_n) = (1/\alpha)u(x). \end{aligned}$$

Thus $K(\cdot, z) - w \geq (1/\alpha)v > 0$. Hence $K(\cdot, z)$ is not equal to its balayage on the complement of U , i.e., U is a fine neighborhood of z . Since $U = U_\varepsilon$ outside of a compact subset of X and the traces on X of Martin neighborhoods of z are also fine

neighborhoods, it follows from the definition of the maximal filter that the maximal filter at z is coarser than the filter of fine neighborhoods at z .

Assume now for the sake of contradiction that $K(\cdot, z)$ is bounded. Then, using the above notation, the function v is also bounded. Since on U the function $G(x_0, \cdot)$ is bounded away from zero, one can find a constant β such that $v \leq \beta \cdot G(x_0, \cdot)$ on U . By extending the function v with the value 0 outside U and taking the upper semicontinuous regularization of the extended function, we obtain a subharmonic function v_1 which is also dominated by $\beta \cdot G(x_0, \cdot)$. Thus $v_1 = 0$, but this contradicts the fact that $v \neq 0$. $\square$

COROLLARY 1. *Assume that any bounded harmonic function on X is constant. Then the harmonic part Γ of the Martin boundary consists of one minimal point z_0 which is not semiregular.*

PROOF. Let μ denote the representing measure of the constant function 1 supported by the minimal points of Δ . Recall that Γ is the support of μ . If Γ contains at least two points, then there exists a measurable subset A of Γ such that $0 < \mu(A) < 1$. The harmonic function $H(\chi_A)$ is bounded, nonzero and not constant since the representing measures supported by the minimal points are unique. Since now $K(\cdot, z_0) = 1$, the rest follows from the last assertion of Theorem 2. $\square$

COROLLARY 2. *For any minimal point z of the Martin boundary and any potential $q \in P_0$, we have $\lim_X (q/K(\cdot, z))(z) = 0$.*

PROOF. The assertion follows from the previous corollary and the fact that for the harmonic space obtained from the existing one by division with $K(\cdot, z)$, the harmonic part of the Martin boundary contains only the single point z which cannot be semiregular. $\square$

COROLLARY 3. *At any polar point x of X , $G(x, x) = +\infty$.*

PROOF. Let $Y = X - \{x\}$. The restriction to Y of the harmonic structure on X and the restriction of G to $Y \times Y$ satisfy the appropriate axioms given above. The Martin compactification $\bar{Y}$ may be identified with $\bar{X}$ so that x is an isolated boundary point. Here $G(\cdot, x)/p(x)$ is the Martin kernel at x provided that we have taken $x_0 \neq x$. By Corollary 2, $\lim_{y \rightarrow x} p(y) \cdot p(x)/G(y, x) = 0$. Thus $G(x, x) = \lim_{y \rightarrow x} G(y, x) = +\infty$. $\square$

Let $\hat{X}$ denote the almost everywhere regular metrizable compactification described in [11] using the reference measure μ_{x_0} . This is also the smallest compactification on which the Dirichlet solution of any function $f \in L$ has a continuous extension to the boundary. Let $\hat{\Delta}$ and $\hat{\Gamma}$ be the boundary and harmonic part of the boundary respectively for $\hat{X}$. Recall that there is a continuous kernel q defined on $X \times \hat{X}$ such that for each $y \in \hat{X}$, $q(\cdot, y)$ is harmonic and for each $x \in X$, $q(x, \cdot)$ is the Radon-Nikodym derivative on $\hat{\Delta}$, $d\mu_x/d\mu_{x_0}$. In particular, $q(x_0, \cdot) \equiv 1$. Also recall that for each $x, y \in X$, $q(x, y) = q(y, x)$. A point $z \in \hat{\Delta}$ is called minimal if $q(\cdot, z)$ is a minimal harmonic function. We denote by $\hat{\Delta}_1$ the set of all minimal points in $\hat{\Delta}$. We recall that each minimal point is regular for the Dirichlet problem and thus is contained in $\hat{\Gamma}$. Also recall that $\hat{\Delta}_1$ is a G_δ set. Let Γ denote the harmonic part of the Martin boundary Δ .

PROPOSITION 5. *Supply the set of positive harmonic functions on X taking the value 1 at x_0 with the topology of uniform convergence on compact sets. We may identify Γ with the space of harmonic functions $\{K(\cdot, y) : y \in \Gamma\}$ and $\hat{\Gamma}$ with the space of harmonic functions $\{q(\cdot, y) : y \in \hat{\Gamma}\}$. Now Γ and $\hat{\Gamma}$ are identical topological spaces.*

PROOF. The homeomorphism between the set Γ (or $\hat{\Gamma}$) and the set $\{K(\cdot, y) : y \in \Gamma\}$ (respectively $\{q(\cdot, y) : y \in \hat{\Gamma}\}$) is well known. Considered as sets of harmonic functions, both Γ and $\hat{\Gamma}$ are the support of the harmonic measure for x_0 on the respective boundaries Δ and $\hat{\Delta}$. In both cases, the harmonic measure for x_0 is exactly the representing measure of 1 that is maximal with respect to the Choquet ordering. Thus Γ and $\hat{\Gamma}$ are equal. $\square$

COROLLARY. *Let μ be a Radon measure on the Martin boundary charging only the minimal points. Denote by h the function represented by μ and let Γ_h be the support of μ . For a positive harmonic function u on X represented by a measure ν the following statements are equivalent:*

- (i) $\nu(\Delta - \Gamma_h) = 0$.
- (ii) *There is a sequence u_n of positive harmonic functions converging uniformly on compact sets to u with u_n/h bounded for each $n \in \mathbb{N}$. In particular, a minimal point $z \in \Gamma_h$ if and only if (ii) holds for $K(\cdot, z)$.*

PROOF. The result follows from [3]. $\square$

Theorem 2 compared the maximal filter at a minimal semiregular point $z \in \Delta$ with the fine filter at z . We now compare the maximal filter at z with the neighborhoods obtained from the topology of $\hat{X}$. Ikegami has shown in [10] that neighborhoods in $\hat{X}$ of minimal points of $\hat{\Gamma}$ are fine neighborhoods except for a set E of harmonic measure 0 (also see [3]). We show here that semiregular points of $\Gamma \cap \Delta_1$ must be in E . Therefore, the set of semiregular points has harmonic measure 0. We note, however, that in an example of Shur [13] there is a point irregular for the Dirichlet problem that has positive harmonic measure.

THEOREM 3. *For any semiregular point $z \in \Gamma \cap \Delta_1$, there is a set A in the maximal filter (and consequently in the fine filter) and a set B which is the trace on X of a neighborhood of z given by the topology from $\hat{X}$ such that A and B are disjoint.*

PROOF. Let μ_z be the measure associated with z introduced in Theorem 1. It follows from that theorem and Corollary 1 of Theorem 2 that μ_z is not unit mass at z . Since L satisfies the Riesz interpolation property there exists by [5] a nonnegative function f in $\overline{D(L)}$ with $f(z) = 0$ and $\int f d\mu_z \neq 0$. The assertion follows since on $\hat{X}$, z is regular whence $H(f)$ converges to 0 at z , but the limit along the maximal filter is $\int f d\mu_z \neq 0$. $\square$

COROLLARY. *The set of semiregular minimal points has harmonic measure 0.*

The following examples shows that the set of semiregular minimal points of Γ may not be empty.

EXAMPLE 1. Let D be the unit disk with boundary T and the usual class of harmonic functions. Then for $x \in D$, $z \in T$, $K(x, z)$ is the Poisson kernel at z .

Let p denote the pointwise infimum of the Green function $G(0, \cdot)$ and the constant function 1. Let z_n be a dense sequence in $T - \{1\}$. For each $n \in \mathbb{N}$, let γ_n be the supremum on the interval $[0, 1]$ of the ratio $K(\cdot, z_n)/p$. Since $z_n \neq 1$, γ_n is finite. Let α_n be a sequence of positive numbers such that $\sum \alpha_n \cdot \gamma_n$ converges. The series $\sum \alpha_n \cdot K(\cdot, z_n)$ converges to a positive harmonic function h on D . We now consider the h -harmonic functions, i.e., the ratios of ordinary harmonic functions divide by h . This is a harmonic class in the sense of Brelot. The Martin boundary for this space is still T , and the representing measure for 1 does not charge the complement of the dense set $\{z_n : n \in \mathbb{N}\}$; the set T equals the harmonic part Γ . We now show that $\limsup_D (p/h)(1) > 0$. Let $\sigma = \sum \alpha_n \cdot \gamma_n$. From the definition of γ_n we have $h/p = \sum \alpha_n \cdot K(\cdot, z_n)/p \leq \sigma$ on $[0, 1]$. Hence $p/h \geq 1/\sigma$ on $[0, 1]$, so $\limsup_D p/h > 0$ at 1.

EXAMPLE 2. Let S denote the spine of Lebesgue in $\mathbb{R}^3$ with the vertex at 0 and let X be the complement of S . By a result of A. Ancona [1], the Martin boundary of X associates one point, which is minimal, to 0. Since the Green function for X has a positive lim sup at 0, 0 is a semiregular point of the Martin boundary which is also a point of Γ since Γ is closed.

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