COMPLEX INTERPOLATING POLYNOMIALS

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ABSTRACT. Let $I_{n,m}(f, z)$ be the unique interpolatory polynomial of degree $\leq 2n - 1$ satisfying the conditions given by (1.1) where the z_{kn} 's are the zeros of the polynomial $z^n - 1$. The object of this paper is to consider the rate of convergence of $I_{n,m}(f, z)$ to $f(z)$ in the L_p norm where $f \in C[|z| \leq 1]$. This problem was initially raised by P. Turán in the case $p = 2$ and in this case the solution was obtained by J. Szabados and A. K. Varma in [7].

Let $f(z)$ be an analytic function in the open unit circle $|z| < 1$ and continuous in $|z| \leq 1$. (This class of functions will be denoted by $C[|z| \leq 1]$.) The main results of this paper can be described as follows.

THEOREM 1. *Let $f(z) \in C[|z| \leq 1]$, and let $I_{n,m}(f, z)$ be the unique interpolatory polynomial of degree $\leq 2n - 1$ satisfying the conditions*

$$(1.1) \quad I_{n,m}(f, z_{kn}) = f(z_{kn}), \quad I_{n,m}^{(m)}(f, z_{kn}) = 0, \quad k = 1, 2, \dots, n,$$

where

$$(1.2) \quad z_{kn} = e^{2\pi i k/n}, \quad k = 1, 2, \dots, n, \quad \theta_{kn} = 2\pi k/n,$$

are the zeros of the polynomial $z^n - 1 = 0$. Then for any fixed $0 < p < \infty$ we have

$$(1.3) \quad \left(\frac{1}{2\pi} \int_{C:|z|=1} |f(z) - I_{n,m}(f, z)|^p |dz| \right)^{1/p} \leq c w_m \left(f, \frac{1}{n} \right)$$

where c is a positive constant depending only on p . Here $w_m(f, \delta)$ is the generalized modulus of continuity of $f(e^{i\theta})$, $0 \leq \theta \leq 2\pi$.

REMARK 1. In the case of $p = 2$, the above theorem reduces to the one obtained earlier [7] in a joint work with J. Szabados (with $w(f, 1/n)$ on the right side of the inequality (1.3)). The method of proof is very different than the one used earlier. Moreover, as mentioned in [7], the solution of Turán's open problem XLVI [8] is indeed a special case of Theorem 1.

We shall also prove the following.

THEOREM 2. *Let $f(z) \in C[|z| \leq 1]$, and let $R_{n,m}(f, z)$ be the unique interpolatory polynomial of degree $\leq 2n - 1$ satisfying the conditions*

$$(1.4) \quad R_{n,m}(f, z_{kn}) = f(z_{kn}), \quad R_{n,m}^{(m)}(f, z_{kn}) = \beta_{kn} \quad (k = 1, 2, \dots, n).$$

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Then we have

$$(1.5) \quad \lim_{n \rightarrow \infty} \left(\frac{1}{2\pi} \int_{C: |z|=1} |f(z) - R_{n,m}(f, z)|^p |dz| \right)^{1/p} = 0, \quad 0 < p < \infty,$$

provided

$$(1.6) \quad \beta_{kn} = o(n^m), \quad k = 1, 2, \dots, n.$$

REMARK 2. From earlier results of Riemenschneider and A. Sharma [4, Theorem 4.1], it follows that if $\beta_{kn} = o(n^m / \log n)$, $f(z) \in C[|z| \leq 1]$ together with some Dini Lipschitz condition then $R_n(f, z)$ converges uniformly to $f(z)$ on $|z| \leq 1$. Thus in the case of L_p convergence of $R_n(f, z)$ to $f(z)$ we have gained more freedom of β_{kn} . We can also drop the Dini Lipschitz condition of $f(z)$ ($z = e^{i\theta}$).

2. Preliminaries. It is known that [3, 4]

$$(2.1) \quad I_{n,m}(f, z) = \sum_{k=1}^n f(z_{kn}) A_{k,n,m}(z),$$

and

$$(2.2) \quad R_{n,m}(f, z) = \sum_{k=1}^n f(z_{kn}) A_{k,n,m}(z) + \sum_{k=1}^n \beta_{knm} B_{k,n,m}(z),$$

where

$$(2.3) \quad A_{k,n,m}(z) = l_{kn}(z) - \frac{1}{n} \sum_{\nu=0}^{n-1} \lambda_{\nu,n,m} z_{kn}^{n-\nu} z^\nu (z^n - 1),$$

$$(2.4) \quad B_{k,n,m}(z) = \frac{(z^n - 1)}{n} \sum_{\nu=0}^{n-1} z_{kn}^{m-\nu} \mu_{\nu,n,m} z^\nu.$$

Here

$$(2.5) \quad l_{kn}(z) = \frac{z_{kn}}{n} \frac{z^n - 1}{z - z_{kn}} = \frac{1}{n} \sum_{\nu=0}^{n-1} \left(\frac{z}{z_{kn}} \right)^\nu,$$

$$(2.6) \quad \lambda_{\nu,n,m} = \frac{(\nu)_m}{(\nu+n)_m - (\nu)_m}, \quad \mu_{\nu,n,m} = \frac{1}{(\nu+n)_m - (\nu)_m},$$

$$(2.7) \quad (\nu)_m = \nu(\nu-1) \cdots (\nu-m+1).$$

We also note from (2.6) and (2.7)

$$(2.8) \quad |\lambda_{j,n,m}| \leq c_1, \quad |\mu_{j,n,m}| \leq c_2/n^m,$$

$$(2.9) \quad |\lambda_{n-1,n,m} - \lambda_{n,n,m}| \leq c_3/n, \quad |\mu_{n-1,n,m} - \mu_{n,n,m}| \leq c_4/n^{m+1},$$

$$(2.10) \quad |\lambda_{j-1,n,m} - 2\lambda_{j,n,m} + \lambda_{j+1,n,m}| \leq c_5/n^2,$$

$$(2.11) \quad |\mu_{j-1,n,m} - 2\mu_{j,n,m} + \mu_{j+1,n,m}| \leq c_6/n^{m+2},$$

where $c_1, c_2, \dots$ are all absolute constants.

Let us denote by

$$(2.12) \quad t_{1,k}(\theta) = 1, \quad t_{\nu,k}(\theta) = 1 + \frac{2}{\nu} \sum_{j=1}^{\nu-1} (\nu-j) \cos j(\theta - \theta_{kn}), \quad 1 \leq \nu \leq n,$$

the well-known Féjer kernel, also $\tilde{t}_{\nu,k}(\theta)$ is the trigonometric polynomial conjugate to $t_{\nu,k}(\theta)$. We set

$$(2.13) \quad p_{1,k}(z) = 1, \quad p_{\nu,k}(z) = 1 + \frac{2}{\nu} \sum_{j=1}^{\nu-1} (\nu - j) \left(\frac{z}{z_{kn}} \right)^j, \quad \nu = 2, 3, \dots$$

Then a simple computation shows that

$$(2.14) \quad p_{\nu,k}(z) = t_{\nu,k}(\theta) + i\tilde{t}_{\nu,k}(\theta), \quad z = e^{i\theta},$$

$$(2.15) \quad 2(z/z_{kn})^\nu = (\nu + 1)p_{\nu+1,k}(z) - 2\nu p_{\nu,k}(z) + (\nu - 1)p_{\nu-1,k}(z)$$

and

$$(2.16) \quad np_{n,k}(z) = (n - 1)p_{n-1,k}(z) - 1 + 2nl_{kn}(z).$$

3. Some lemmas. The following lemmas are needed for the proof of our theorems.

LEMMA 3.1. *Let us denote by*

$$(3.1) \quad g_\nu(z) = \frac{1}{n} \sum_{k=1}^n \alpha_{kn} p_{\nu,k}(z), \quad \alpha_{kn} = u_{kn} + iv_{kn}, |\alpha_{kn}| \leq c_1,$$

where $p_{\nu,k}(z)$ is defined by (2.13). Then

$$(3.2) \quad \|g_\nu\|_{L_p} \leq c_1 c_2, \quad 1 < p < \infty,$$

where c_2 is an absolute constant depending only on p .

PROOF. Since $|\alpha_{kn}| \leq c_1$ it follows that

$$(3.3) \quad |u_{kn}| \leq c_1, \quad |v_{kn}| \leq c_1.$$

Also, it is known that

$$(3.4) \quad t_{\nu,k}(\theta) \geq 0, \quad \frac{1}{n} \sum_{k=1}^n t_{\nu,k}(\theta) = 1.$$

Thus it is easy to conclude that

$$(3.5) \quad \left| \frac{1}{n} \sum_{k=1}^n u_{kn} t_{\nu,k}(\theta) \right| \leq c_1, \quad \left| \frac{1}{n} \sum_{k=1}^n v_{kn} t_{\nu,k}(\theta) \right| \leq c_1.$$

Now by using a well-known theorem of Riesz [9, Theorem 2.4, p. 253], (3.5) and the formula

$$(3.6) \quad \begin{aligned} g_\nu(z) &= \frac{1}{n} \sum_{k=1}^n u_{kn} t_{\nu,k}(\theta) - \frac{1}{n} \sum_{k=1}^n v_{kn} \tilde{t}_{\nu,k}(\theta) \\ &\quad + \frac{i}{n} \sum_{k=1}^n u_{kn} \tilde{t}_{\nu,k}(\theta) + \frac{i}{n} \sum_{k=1}^n v_{kn} t_{\nu,k}(\theta). \end{aligned}$$

We immediately obtain (3.2).

LEMMA 3.2. Let $f(z) \in C[|z| \leq 1]$. Then $I_m(f, z)$ as defined by (2.1), (2.3) satisfies the estimate

$$(3.7) \quad \|I_m(f, z)\|_{L_p} \leq c_p \|f\|_{L_\infty}$$

where c_p depends only on p . Also, let

$$(3.8) \quad Q_n(z) = \sum_{k=1}^n v_{knm} B_{k,n,m}(z)$$

where

$$(3.9) \quad |v_{knm}| \leq \varepsilon_{nm}.$$

Then we have

$$(3.10) \quad \|Q_n(z)\|_{L_p} \leq \varepsilon_{nm} c_4(p) n^{-m}.$$

Here $B_{k,n,m}(z)$ is defined by (2.4), (2.6).

From (2.15), (2.16), (2.3) and some simple calculations it follows that

$$(3.11) \quad A_{k,n,m}(z) = \frac{z^n - 1}{2n} \lambda_{n-1,n,m} + l_{k,n}(z)(1 - (z^n - 1)\lambda_{n-1,n,m}) \\ - \frac{z^n - 1}{2n} \left\{ \sum_{j=1}^{n-1} (\lambda_{j-1,n,m} - 2\lambda_{j,n,m} + \lambda_{j+1,n,m}) j p_{j,k}(z) \right. \\ \left. + (\lambda_{n-1,n,m} - \lambda_{n,n,m})(n-1)p_{n-1,k}(z) \right\}.$$

Also as a consequence of the L_p convergence of the Lagrange interpolation process based on the n th roots of unity (Marcinkiewicz Theorem). Chapter 10 of [9], it follows that

$$(3.12) \quad \left\| \sum_{k=1}^n f(z_{kn}) l_{kn}(z) \right\|_{L_p} \leq c_7 \|f\|_{L_\infty}, \quad 1 < p < \infty.$$

Now, on using (2.8), (2.9), (2.10), (3.11), (3.12), estimate (3.7) follows. Proof of (3.10) is on similar lines; there one uses (3.9), (3.8), (2.8), (2.9), (2.11), (3.12) and the following representation of the fundamental polynomials of the second kind. We omit the details.

$$(3.13) \quad B_{k,n,m}(z) = \frac{z^n - 1}{2n} z_{kn}^m \left[-\mu_{0,n,m} + (2nl_{kn}(z) + 1)\mu_{n-1,n,m} \right. \\ \left. + (n-1)p_{n-1,k}(z)(\mu_{n-1,n,m} - \mu_{n,n,m}) \right. \\ \left. + \sum_{\nu=1}^{n-1} \nu(\mu_{\nu-1,n,m} - 2\mu_{\nu,n,m} + \mu_{\nu+1,n,m})p_{\nu,k}(z) \right].$$

In the proof of our theorems the next lemma [10] plays an important role.

LEMMA 3.3. Let $f(z) \in C[|z| \leq 1]$. Let $w_m(f, \delta)$ be the generalized modulus of continuity of $f(e^{i\theta})$, $0 \leq \theta \leq 2\pi$. Then there exists a polynomial $F_n(z)$ of degree $\leq n-1$ such that for $|z| \leq 1$ we have

$$(3.14) \quad |f(z) - F_n(z)| \leq c_8 w_m(f, 1/n)$$

and

$$(3.15) \quad |F_n^{(m)}(z)| \leq c_9 n^m w_m(f, 1/n).$$

4. Proof of Theorem 1. It is enough to prove Theorem 1 for $1 < p$. (See [9, p. 25, Theorem 10.12].) Let $f(z) \in C[|z| \leq 1]$. We denote by $F_n(z)$ the polynomial of degree $\leq n-1$ satisfying (3.14) and (3.15). Then from (2.1)

$$(4.1) \quad \begin{aligned} f(z) - I_{n,m}(f, z) &= f(z) - F_n(z) + F_n(z) - I_{n,m}(F_n, z) \\ &\quad + I_{n,m}(F_n - f, z) = f(z) - F_n(z) + I_{n,m}(F_n - f, z) \\ &\quad + \sum_{k=1}^n F_n^{(m)}(z_{kn}) B_{k,n,m}(z), \end{aligned}$$

On using (3.14), (3.7) we have ($p > 1$)

$$(4.2) \quad \|I_{n,m}(F_n - f, z)\|_{L_p} \leq c_p \|F_n - f\|_{L_\infty} \leq c_p c_8 w_m(f, 1/n).$$

Also, from (3.15), (3.10), (3.9) we obtain

$$(4.3) \quad \left\| \sum_{k=1}^n F_n^{(m)}(z_{kn}) B_{k,n,m}(z) \right\|_{L_p} \leq c_9 n^m w_m(f, 1/n) c_4(p) n^{-m} \leq c_{10} w_m(f, 1/n).$$

Now, from (4.1), (4.2), (4.3) and (3.14) we have (1.3). This proves Theorem 1 for $1 < p < \infty$.

The proof of Theorem 2 is on the same lines so we omit the details.

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BIBLIOGRAPHY

1. A. S. Cavaretta, A. Sharma and R. S. Varga, *Hermite Birkhoff interpolation in the n th roots of unity*, Trans. Amer. Math. Soc. **259** (1980), 621–628.
2. O. Kis, *Notes on interpolation*, Acta Math. Acad. Sci. Hungar. **11** (1960), 49–64.
3. G. G. Lorentz, K. Jetter and S. D. Riemenschneider, *Birkhoff interpolation*, Encyclopedia of Mathematics and its Application, Vol. 19, (see Chapter XII), 1983.
4. S. D. Riemenschneider and A. Sharma, *Birkhoff interpolation at the n th roots of unity*, Canad. J. Math. **33** (1981), 362–371.

5. A. Sharma and P. Vertesi, *Mean convergence of interpolation in roots of unity*, SIAM J. Math. Anal. **14** (1983), 800–806.
6. J. Szabados, *On some convergent interpolatory polynomials*, Fourier Analysis and Approximation Theory (G. Alexits and P. Turan, eds.), vol. II, North-Holland, 1978, pp. 805–816.
7. J. Szabados and A. K. Varma, *On an open problem of P. Turan concerning Birkhoff interpolation based on the roots of unity*, J. Approximation Theory **47** (1986), 255–264.
8. P. Turán, *On some open problems of approximation theory*, J. Approximation Theory **29** (1980), 23–85.
9. A. Zygmund, *Trigonometric series*, vols. I, II, Cambridge Univ. Press, 1968.
10. A. Sharma and J. Szabados, *Convergence rates for some lacunary interpolators on their roots of unity* (preprint April 1987).

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