SOLUTIONS WITH COMPACT SUPPORT OF THE POROUS MEDIUM EQUATION IN ARBITRARY DIMENSIONS

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ABSTRACT. Compactness of the support is discussed of a solution u to the Cauchy problem for the porous medium equation $u_t = \Delta\phi(u)$, $t > 0$, in R^N of arbitrary dimension $N \geq 1$, where ϕ is a nondecreasing function on R^1 . It is shown that if $u(0, x) = 0$ for $|x| \geq R$, $R > 0$, then for all $t \geq 0$

$$u(t, x) = 0 \quad \text{a.e. } |x| \geq R + Ct^{1/2}$$

with a constant C depending on ϕ and $u(0, \cdot)$.

The result is well known when $N = 1$, but the study for $N > 1$ has somehow been neglected.

1. Statement of the result. Consider the Cauchy problem for the porous medium equation in R^N for $N \geq 1$:

$$\begin{aligned} (1) \quad & u_t = \Delta\phi(u), \quad t > 0, \quad x \in R^N; \\ (2) \quad & u(0, x) = u_0(x), \quad x \in R^N. \end{aligned}$$

In the case of $\phi(r) = r$, the solution is given by

$$(4\pi t)^{-N/2} \int_{R^N} e^{-|x-y|^2/4t} u_0(y) dy$$

and is shown to spread out in R^N for any $t > 0$ unless $u_0(x) = 0$ for a.a. $x \in R^N$. But the situation is different if $\phi(r) = |r|^{m-1}r$, $m > 1$, for example. The purpose of this paper is to show under more general assumptions on ϕ that if

$$(3) \quad u_0 \in L^\infty(R^N); \quad u_0(x) = 0 \quad \text{for } |x| \geq R, \quad R > 0,$$

then the solution (in a generalized sense) $u(t, x)$ of (1) and (2) stays compactly supported in R^N forever: For all $t \geq 0$

$$u(t, x) = 0 \quad \text{a.e. } |x| \geq R + Ct^{1/2}$$

with a constant $C > 0$ depending on ϕ and $\|u_0\|_\infty$.

This kind of phenomenon is well known when $N = 1$: Even the lateral boundary, called the interface, of the support of $u(t, x)$ in R^N has been discussed extensively together with its behavior (see e.g. [4, 5]). However, the study in the case of $N > 1$ has somehow been neglected. Within the Cauchy problem (1)–(2) little is known about compact support of the solution (cf. Diaz Diaz [3] and Schatzman [6]).

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In this paper we shall deal with solutions of (1) in $D'((0, \infty) \times R^N)$ (in the sense of distributions on $(0, \infty) \times R^N$) under the following conditions on ϕ :

(4) ϕ is a nondecreasing continuous function on R^1 with $\phi(0) = 0$;

(5) the range of ϕ coincides with R^1 , and for each $r \in R^1$ the integral $\int_0^r 1/\phi^{-1}(s) ds$ exists.

Based on the work [1], Brézis and Crandall [2] established that, for each $u_0 \in L^1(R^N) \cap L^\infty(R^N)$, there is a unique solution u of (1)–(2) in $D'((0, \infty) \times R^N)$ such that

$$(6) \quad u \in C([0, \infty); L^1(R^N)) \cap L^\infty((0, \infty) \times R^N).$$

It is suggested in the paper [2, (1.20)] that for each $T > 0$,

$$(7) \quad u_0 \in L^1(R^N) \cap L^\infty(R^N); \quad f \in L^1(0, T; L^1(R^N) \cap L^\infty(R^N)),$$

there exists a unique solution u in $D'((0, T) \times R^N)$ of

$$(1)' \quad u_t = \Delta \phi(u) + f(t, x), \quad 0 < t < T, \quad x \in R^N,$$

and (2) such that

$$(6)' \quad u \in C([0, T]; L^1(R^N)) \cap L^\infty((0, T) \times R^N).$$

The main result is formulated as follows:

THEOREM. *Under the conditions (3)–(5), the solution u in $D'((0, \infty) \times R^N)$ of (1)–(2) with (6) satisfies, for all $t \geq 0$,*

$$u(t, x) = 0 \quad \text{a.e. } |x| \geq R + 2C_0 t^{1/2}$$

with

$$C_0 = \left\{ \int_0^{\phi(\|u_0\|_\infty)} \frac{1}{\phi^{-1}(s)} ds \right\}^{1/2}.$$

The theorem coincides with a result for $N = 1$ of Knerr [4, Theorem 8.4] and is similar to a result of Diaz Diaz [3, Theorem I] for an initial-boundary value problem.

2. Proof of the main result. The difficulty in this paper is that no exact solution of (1) can be found, for N is arbitrary and the explicit form of $\phi(r)$ is not necessarily given. So the main task is to show that the function $v = v(t, x)$, determined by

$$\int_0^{\phi(v)} \frac{1}{\phi^{-1}(s)} ds = \max \left(0, a - \frac{|x|^2}{4(t+b)} \right),$$

where a and b are positive constants, is compactly supported and a solution in $D'((0, T) \times R^N)$ of (1)' with a nonnegative and "small" function $f(t, x)$. If it is done, the proof is carried out as usual by using the comparison principle (Lemma 2) for solutions of (1)'–(2).

The function $y(r)$, defined under (4) and (5) by

$$y(r) = \phi^{-1}(z^{-1}(r)) \quad \text{with} \quad z(r) = \int_0^r \frac{1}{\phi^{-1}(s)} ds,$$

is a nondecreasing continuous function on $[0, \infty)$ with

$$(8) \quad (d/dr)\phi(y(r)) = y(r) \quad \text{and} \quad y(0) = 0,$$

and our function v can be written as

$$v(t, x) = y((a - |x|^2/4(t+b))^+),$$

where $r^+ = \max(0, r)$ for $r \in R^1$. From this we see that v is nonnegative valued, and satisfies

$$(9) \quad v(t, x) = 0 \quad \text{for } t \geq 0 \quad \text{and} \quad |x|^2 \geq 4a(t+b)$$

and, by Lebesgue's theorem, (6)' for each fixed $T > 0$.

We shall show that v is a solution in $D'((0, T) \times R^N)$ of (1)'-(2) with $(u_0; f)$ replaced by $(v_0; g)$:

$$\begin{cases} v_0(x) = y((a - |x|^2/4b)^+), \\ g(t, x) = y((a - |x|^2/4(t+b))^+)N/2(t+b), \end{cases}$$

for which the condition (7) is valid.

LEMMA 1. For $t \geq 0$ and $x \in R^N$, v satisfies

$$(10) \quad \sum_{i=1}^N \frac{\partial}{\partial x_i} \int_0^t \frac{\partial}{\partial x_i} \phi(v) dr = [v(r, x)]_0^t - \int_0^t g(r, x) dr.$$

PROOF. The proof is done by an elementary calculus. We have only to note (8) to see that

$$\begin{aligned} \int_0^t \frac{\partial}{\partial x_i} \phi(v) dr = & -\frac{2x_i}{|x|^2} \left\{ \left[(r+b)\phi \left(y \left(\left(a - \frac{|x|^2}{4(r+b)} \right)^+ \right) \right) \right]_0^t \right. \\ & \left. - \int_0^t \phi \left(y \left(\left(a - \frac{|x|^2}{4(r+b)} \right)^+ \right) \right) dr \right\} \end{aligned}$$

if $|x| \neq 0$, and equals 0 if $|x| = 0$. Q.E.D.

Thus, multiplication by ψ_t with $\psi \in C_0^\infty((0, T) \times R^N)$ of (10) and integration over $(0, T) \times R^N$ gives

$$\int_0^T \int_{R^N} (v\psi_t + \phi(v)\Delta\psi + g(t, x)\psi) dx dt = 0.$$

We are now in a position to examine the support in R^N of the solution $u(t, x)$ in comparison with that of $v(t, x)$.

LEMMA 2. Take $T > 0$ and let $(u_0; f)$ and $(\hat{u}_0; \hat{f})$ satisfy the condition (7). Then the corresponding solutions u and $\hat{u}$ in $D'((0, T) \times R^N)$ of (1)'-(2) with (6)' satisfy

$$\|(u(t, \cdot) - \hat{u}(t, \cdot))^+\|_1 \leq \|(u_0 - \hat{u}_0)^+\|_1 + \int_0^t \|(f(r, \cdot) - \hat{f}(r, \cdot))^+\|_1 dr$$

for $0 \leq t \leq T$.

PROOF. The lemma is a consequence of the property of the operator $\Delta\phi$ in $(L^1(R^N), \|\cdot\|_1)$:

$$\int_{R^N} \operatorname{sgn}^+(u-v) \cdot (\Delta\phi(u) - \Delta\phi(v)) dx \leq 0,$$

where

$$\operatorname{sgn}^+ r = \begin{cases} 1 & (r > 0), \\ 0 & (r \leq 0). \end{cases}$$

The proof is standard and might be omitted (see [3, §3; 7]). Q.E.D.

PROOF OF THE THEOREM. Assume in addition that $u_0(x)$ is nonnegative for all $x \in R^N$ without loss of generality. Let a and b satisfy $a > R^2/4b$ and

$$(11) \quad \|u_0\|_\infty = y(a - R^2/4b).$$

Then $y((a - |x|^2/4b)^+)$ is not smaller than $u_0(x)$ for a.a. $|x| < R$ and obviously for $|x| \geq R$, and hence

$$0 \leq u_0(x) \leq v_0(x) \quad \text{for a.a. } x \in R^N.$$

Since $g(t, x)$ is nonnegative, we can apply Lemma 2 to the solutions $u(t, x)$ and $v(t, x)$ with data $(u_0; 0)$ and $(v_0; g)$ respectively to obtain

$$0 \leq u(t, x) \leq v(t, x) \quad \text{for a.a. } x \in R^N$$

and for all $0 \leq t \leq T$ with each fixed $T > 0$.

Recalling (9) with (11), we see that $u(t, x)$ vanishes for all $t \geq 0$ and a.a. $x \in R^N$ such that

$$|x|^2 \geq 4(t+b)\{R^2/4b + z(\phi(\|u_0\|_\infty))\}, \quad b > 0.$$

We have now only to find out the envelope. Choosing $b = (R/2)\{t/z(\phi(\|u_0\|_\infty))\}^{1/2}$ for $t > 0$, we can conclude

$$u(t, x) = 0 \quad \text{a.e. } |x| \geq R + 2\{z(\phi(\|u_0\|_\infty))t\}^{1/2}$$

for all $t \geq 0$. Q.E.D.

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