A RUNGE THEOREM FOR HARMONIC FUNCTIONS ON CLOSED SUBSETS OF RIEMANN SURFACES

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ABSTRACT. Let F be a closed subset of a Riemann surface Ω . It is shown that every function which is harmonic on a neighborhood of F can be uniformly approximated on F by functions which are harmonic on Ω except for poles.

1. Introduction. In this paper we prove a theorem of Runge type for uniform approximation by harmonic functions on closed subsets of Riemann surfaces. If Ω is a Riemann surface, a continuous function $h: \Omega \rightarrow [-\infty, \infty]$ will be said to be *essentially harmonic* provided that it is harmonic except for logarithmic singularities; that is, provided that for each point $p \in \Omega$ there exist a local coordinate map z near p , a real number c , and a harmonic function w near p such that $h(q) \equiv w(q) + c \log |z(q) - z(p)|$ in some neighborhood of p .

THEOREM. *Let F be a closed subset of a Riemann surface Ω , and let f be a harmonic function on an open set containing F . If $\varepsilon > 0$, then there exists an essentially harmonic function h on Ω such that $\sup_F |f - h| < \varepsilon$.*

The analogous assertion for holomorphic functions would be false. In fact, from work of Gauthier and Hengartner [4] one sees that there exists a Riemann surface Ω , a closed set $F \subset \Omega$, and a holomorphic function f on a neighborhood of F , with the following property: if h is holomorphic on Ω except for isolated singularities, then $\sup_F |f - h| > 1$. This was pointed out to the author by Gauthier.

Special cases of the theorem follow from work of Lax, Malgrange, Pfluger, Boivin, Gauthier, Goldstein and Ow; for a discussion of this work we refer to [5, Chapitre 2] and [2]. In particular, we note that the theorem is proved in [5, Corollaire 2.3.9] under the additional assumption that F is "essentially of finite genus" (for terminology see [5, p. 29]); and the theorem is proved in [5, Théorème 2.3.5] if "essentially harmonic" is replaced by "harmonic on the complement of a discrete subset of Ω ." In a forthcoming joint paper with P. M. Gauthier the results proved here will be combined with arguments concerning balayage of poles; in particular, we will show that under appropriate additional hypotheses one can obtain approximating functions h which are harmonic on all of Ω .

We recall some well known results concerning Riemann surfaces in §2. The proof of the theorem is then given in §3.

2. Notation and preliminaries. We set $B_r = \{z \in \mathbb{C} : |z| < r\}$. The notation $f: A \twoheadrightarrow B$ indicates that the function $f: A \rightarrow B$ is onto.

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A *Riemann surface* is a connected complex-analytic manifold of complex dimension one. If Ω is a Riemann surface, a 2-form α on Ω assigns a function α_z for each coordinate map $z = x + iy$, such that on overlapping coordinate regions we have $\alpha_\zeta = \alpha_z |dz/d\zeta|^2$; we write $\alpha = \alpha_z dx \wedge dy = i\alpha_z dz \wedge d\bar{z}/2$. If α is a continuous 2-form, corresponding to a function α_z for each local coordinate z , then $|\alpha|$ denotes the continuous 2-form which assigns the function $|\alpha_z|$ to each local coordinate z .

Every Riemann surface is either hyperbolic or parabolic (see [6]). In the remainder of this section we summarize some facts associated with this classification.

If Ω is a hyperbolic Riemann surface, then there exists a *Green's kernel* for Ω , which is the unique mapping $g: \Omega \times \Omega \rightarrow (0, \infty]$ with the following properties [6, Chapter V, §2, 8].

(G1) For each point $q \in \Omega$, the function $\Omega \setminus \{q\} \ni p \rightarrow g(p, q)$ is harmonic.

(G2) If ω is a parametric disc for Ω , and $z: \omega \rightarrow B_1$ is a conformal homeomorphism, then we may write $g(p, q) = -\log |z(p) - z(q)| + v(p, q)$ for $p, q \in \omega$, where $v: \omega \times \omega \rightarrow \mathbf{R}$ is (jointly) continuous, and, for each point $a \in \omega$, the functions $\omega \ni p \rightarrow v(p, a)$ and $\omega \ni q \rightarrow v(a, q)$ are harmonic.

(G3) If $q \in \Omega$, and u is any bounded harmonic function on Ω with $g(\cdot, q) \geq u \geq 0$, then $u \equiv 0$.

(G4) $g(p, q) = g(q, p)$ for all points $p, q \in \Omega$.

If Ω is an open parabolic Riemann surface, there is an *Evans kernel* for Ω , which is a function $e: \Omega \times \Omega \rightarrow (-\infty, \infty]$ with the following properties [6, Chapter V, §3, 13] (these properties are analogous to those for $e(p, q) \equiv -\log |p - q|$ when $\Omega = \mathbf{C}$):

(E1) for each point $q \in \Omega$, the function $\Omega \setminus \{q\} \ni p \rightarrow e(p, q)$ is harmonic.

(E2) if ω is a regular subregion of Ω , we may write $e(p, q) = g_\omega(p, q) + v(p, q)$ for $p, q \in \omega$, where g_ω is the Green's function for ω , the function $v: \omega \times \omega \rightarrow \mathbf{R}$ is (jointly) continuous, and, for each point $a \in \omega$, the functions $\omega \ni p \rightarrow v(p, a)$ and $\omega \ni q \rightarrow v(a, q)$ are harmonic.

(E3) for each point $q \in \Omega$, there is a compact set $K \subset \Omega$ such that $e(p, q) \leq 0$ for $p \in \Omega \setminus K$.

(E4) if $q \in \Omega$ and $q' \in \Omega$, then there is a compact set $K = K(q, q')$ such that the function $\Omega \setminus K \ni p \rightarrow e(p, q') - e(p, q)$ is bounded.

(E5) $e(p, q) = e(q, p)$ for all points $p, q \in \Omega$.

We will need the following *maximum principle* for parabolic Riemann surfaces: if Ω is an open parabolic Riemann surface, $\omega \subset\subset \Omega$ is a regular subregion, and $u: \Omega \setminus \omega \rightarrow \mathbf{R}$ is a bounded continuous function which is harmonic on $\Omega \setminus \bar{\omega}$ and is ≤ 0 on $\partial\omega$, then $u \leq 0$ on $\Omega \setminus \omega$. (See [6, Chapter V, §2, 7E].)

3. Proof of the theorem. If Ω is compact, then F is compact, and the theorem is well known [5, Théorème 2.1.8].

For the rest of the paper we assume that Ω is an open Riemann surface. We let $E: \Omega \times \Omega \rightarrow (-\infty, \infty]$ denote the Green's kernel if Ω is hyperbolic, and the Evans kernel if Ω is parabolic. The operator $\partial\bar{\partial}$ defines a mapping from C^∞ functions on Ω to 2-forms on Ω ; the singularity of $E(\cdot, q)$ at q has been chosen so that $E(\cdot, q)/\pi i$ may be used as a type of fundamental solution for this operator, as stated in the following lemma.

LEMMA 1. *Let α be a smooth 2-form on Ω having compact support. Then the function*

$$u(p) = \frac{1}{\pi i} \int E(p, q) \alpha(q), \quad p \in \Omega,$$

is smooth and satisfies $\partial\bar{\partial}u = \alpha$.

Since results of this type are well known, we will only outline the proof. By use of a partition-of-unity, the proof may be reduced to the case where the support of α is a compact subset of a parametric disk $\omega \subset\subset \Omega$. In this case the desired properties of u on ω follow from the fact that $(2/\pi) \log |z|$ is a fundamental solution for the operator $\partial^2/\partial z \partial \bar{z}$ in $\mathbb{C}$; and the desired properties of u in a neighborhood of $\Omega \setminus \omega$ follow from the fact that for each point $q \in \Omega$ the function $E(\cdot, q)$ is harmonic on $\Omega \setminus \{q\}$.

LEMMA 2. *Let D be an open subset of Ω , and $q \in D$. If $\varepsilon > 0$, then there exists a parametric disc N such that $q \in N \subset\subset D$ and $|E(p, q) - E(p, q_1)| \leq \varepsilon$ for $p \in \Omega \setminus D$, $q_1 \in N$.*

PROOF. We may assume without loss of generality that there is an open set ω with $D \subset\subset \omega \subset\subset \Omega$ and a conformal homeomorphism $h: \omega \rightarrow B_1$ such that $h(q) = 0$ and the image of D is B_ρ , $\rho < 1$. Since the mapping $\partial B_\rho \times \overline{B_{\rho/2}} \ni (z, \zeta) \rightarrow E(h^{-1}(z), h^{-1}(\zeta))$ is uniformly continuous, we may select a parametric disc N such that $q \in N \subset\subset D$ and

$$(1) \quad |E(p, q) - E(p, q_1)| \leq \varepsilon/3 \quad \text{for } p \in \partial D, \quad q_1 \in \overline{N}.$$

To complete the proof of the lemma we consider separately the case where Ω is parabolic and the case where Ω is hyperbolic.

Suppose first that Ω is parabolic, and let $q_1 \in N$ be fixed. It follows easily from property E4 in §2 that the function $\Omega \setminus D \ni p \rightarrow E(p, q) - E(p, q_1)$ is bounded. From (1) and the maximum principle for parabolic surfaces we conclude that $|E(p, q) - E(p, q_1)| \leq \varepsilon/3 < \varepsilon$ for all $p \in \Omega \setminus D$. This proves Lemma 2 in case Ω is parabolic.

For the rest of the proof of Lemma 2 we assume that Ω is hyperbolic. Let $D \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \cdots$, where each Ω_j is a regular subregion of Ω and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Let g_j be the Green's function for Ω_j ; from [6, Chapter V, §2, 8A] it follows that

$$(2) \quad g_j \uparrow E.$$

(In the proof that follows we will combine (2) with the maximum principle for regular subregions, a technique used in [5, pp. 72–73].) From (2) and Dini's theorem we conclude that $g_j \uparrow E$ uniformly (as real-valued functions) on the compact set $(\partial D) \times \overline{N}$, since the limit E is real-valued and continuous there. We thus see that there exists an index J such that for $p \in \partial D$ and $j \geq J$ we have

$$|g_j(p, q_1) - E(p, q_1)| \leq \varepsilon/3, \quad q_1 \in \overline{N},$$

and hence we have

$$\begin{aligned} |g_j(p, q) - g_j(p, q_1)| &\leq |g_j(p, q) - E(p, q)| + |E(p, q) - E(p, q_1)| \\ &\quad + |E(p, q_1) - g_j(p, q_1)| \\ (3) \quad &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \quad q_1 \in \overline{N}. \end{aligned}$$

We now claim that for each fixed integer $j \geq J$ we have

$$|g_j(p, q) - g_j(p, q_1)| \leq \varepsilon \quad \text{if } p \in \Omega_j \setminus D \text{ and } q_1 \in N.$$

To prove this claim, we fix $q_1 \in N$, and consider the function

$$G(p) \equiv g_j(p, q) - g_j(p, q_1), \quad p \in \overline{\Omega_j} \setminus D.$$

We have $G \equiv 0$ on $\partial\Omega_j$, and from (3) we see that $|G| \leq \varepsilon$ on ∂D ; from the usual maximum principle for harmonic functions it follows that $|G| \leq \varepsilon$ on $\overline{\Omega_j} \setminus D$, which proves the claim.

The lemma now follows by taking fixed points $p \in \Omega \setminus D$ and $q_1 \in N$, and letting $j \rightarrow \infty$ in the claim.

We now complete the proof of the theorem stated in §1. The proof makes use of the Mittag-Leffler technique, and should be compared with the proof of [2, Théorème 1]. Let $N \supset F$ be open, and $f: N \rightarrow \mathbf{R}$ be harmonic. There exists a function $\chi \in C^\infty(\Omega)$ such that $\chi \equiv 1$ on an open neighborhood of F , and $\chi \equiv 0$ on an open neighborhood of $\Omega \setminus N$ [7, 2.2.14]. Let $u = \chi f$, with the understanding that $u \equiv 0$ on $\Omega \setminus N$. Let β be the C^∞ 2-form $\partial\bar{\partial}u$, so that $\text{supp } \beta$ is a closed subset of Ω which does not intersect F .

It is possible to find a sequence $\{K_j\}_{j=1}^\infty$ of compact subsets of Ω such that $\Omega = \bigcup_{j=1}^\infty K_j$ and, for each integer $j \geq 1$, $\Omega \setminus K_j$ has no relatively compact connected components and $K_j \subset \text{int } K_{j+1}$; this is easily proved using [7, 3.10.3] and the fact that a Riemann surface has a second countable topology. For each $j \geq 1$ we let $\psi_j \in C_0^\infty(\text{int } K_{j+1})$ be identically equal to one on a neighborhood of K_j . We define $\Omega_1 = \text{int } K_2$, $\Omega_j = (\text{int } K_{j+1}) \setminus K_{j-1}$ for $j \geq 2$, $\varphi_1 = \psi_1$, and $\varphi_j = \psi_j - \psi_{j-1}$ for $j \geq 2$. Then $\text{supp } \varphi_j \subset \Omega_j$ for $j \geq 2$, and $\sum_{j=1}^k \varphi_j \equiv \psi_k \equiv 1$ near K_k for $k \geq 1$. From Lemma 1 we see that for each $j \geq 1$, the function

$$u_j(p) \equiv \frac{1}{\pi i} \int E(p, q) \varphi_j(q) \beta(q), \quad p \in \Omega,$$

is smooth and satisfies $\partial\bar{\partial}u_j = \varphi_j\beta$. In particular, for each $k \geq 1$ we have

$$\partial\bar{\partial} \left[u - \sum_{j=1}^k u_j \right] = \beta - \beta \sum_{j=1}^k \varphi_j = 0 \quad \text{near } K_k.$$

For each $k \geq 1$, the set K_k has a fundamental system of open neighborhoods S with the property that there are no compact connected components of $\Omega \setminus S$ [7, 3.10.6]; thus we may apply the Lax-Malgrange theorem [7, 3.10.7] inductively to see that for each $k \geq 1$ there exists a harmonic function w_k on Ω such that $\sup_{K_k} |u - \sum_{j=1}^k [u_j - w_j]| < 1/k$. We conclude that $u = \sum_{j=1}^\infty [u_j - w_j]$ on Ω .

Now let $\varepsilon > 0$ be arbitrary. Fix an integer $j \geq 1$. If $m_j \equiv \int |\varphi_j\beta| > 0$, we conclude from Lemma 2 that for each point $q \in \text{supp}(\varphi_j\beta)$ there exists a parametric disc N_q such that $q \in N_q \subset \subset (\Omega \setminus F) \cap \Omega_j$ and

$$|E(p, q) - E(p, s)| \leq \pi\varepsilon/m_j 2^{j+1} \quad \text{for } s \in N_q, p \in F \cup (\Omega \setminus \Omega_j).$$

Since $\text{supp}(\varphi_j\beta)$ is compact, we can find a finite set $\mathcal{A}_j \subset \text{supp}(\varphi_j\beta)$ such that $\text{supp}(\varphi_j\beta) \subset \bigcup_{q \in \mathcal{A}_j} N_q$. Let $\{\eta_q\}_{q \in \mathcal{A}_j}$ be a family of nonnegative functions in

$C_0^\infty(\Omega)$ such that $\text{supp } \eta_q \subset N_q$ for each $q \in \mathcal{A}_j$ and $\sum_{q \in \mathcal{A}_j} \eta_q \equiv 1$ on a neighborhood of $\text{supp}(\varphi_j \beta)$. We then define

$$v_j(p) = \frac{1}{\pi i} \sum_{q \in \mathcal{A}_j} E(p, q) \int \eta_q(s) \varphi_j(s) \beta(s),$$

and note that for each $p \in F \cup (\Omega \setminus \Omega_j)$ we have

$$(4) \quad \begin{aligned} |u_j(p) - v_j(p)| &= \left| \frac{1}{\pi i} \sum_{q \in \mathcal{A}_j} \int [E(p, s) - E(p, q)] \eta_q(s) \varphi_j(s) \beta(s) \right| \\ &\leq \frac{\varepsilon}{m_j 2^{j+1}} \int \left[\sum_{q \in \mathcal{A}_j} \eta_q(s) \right] |\varphi_j(s) \beta(s)| = \frac{\varepsilon}{m_j 2^{j+1}} \int |\varphi_j \beta| = \frac{\varepsilon}{2^{j+1}}. \end{aligned}$$

If $\int |\varphi_j \beta| = 0$, then $u_j \equiv 0$; we then let $\mathcal{A}_j$ be the empty set and define $v_j \equiv 0$.

It follows that $h \equiv \sum_{j=1}^\infty [v_j - w_j]$ is well defined and finite at all points of Ω which are in the complement of the discrete set $\bigcup_{j=1}^\infty \mathcal{A}_j$. In fact, for each compact set $K \subset \Omega$ there is an index J such that v_j is (finite and) harmonic on K for all $j \geq J$ and $\sum_{j=J}^\infty [v_j - w_j]$ is uniformly convergent on K ; in particular, $\sum_{j=J}^\infty [v_j - w_j]$ is harmonic on $\text{int } K$. It follows that h is essentially harmonic on Ω . From (4) we also conclude that for each $p \in F$, $h(p)$ is finite and

$$\begin{aligned} |f(p) - h(p)| &= |u(p) - h(p)| \\ &= \left| \sum_{j=1}^\infty [u_j(p) - w_j(p)] - [v_j(p) - w_j(p)] \right| \\ &= \sum_{j=1}^\infty |u_j(p) - v_j(p)| \leq \sum_{j=1}^\infty \frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This completes the proof of the theorem.

REFERENCES

1. A. Boivin and P. M. Gauthier, *Approximation harmonique sur les surfaces de Riemann*, Canad. J. Math. **36** (1984), 1–8.
2. A. Dufresnoy, P. M. Gauthier, and W. H. Ow, *Uniform approximation on closed sets by solutions of elliptic partial differential equations*, Complex Variables **6** (1986), 235–247.
3. P. M. Gauthier, M. Goldstein, and W. H. Ow, *Uniform approximation on unbounded sets by harmonic functions with logarithmic singularities*, Trans. Amer. Math. Soc. **261** (1980), 169–183.
4. P. M. Gauthier and W. Hengartner, *Uniform approximation on closed sets by functions analytic on a Riemann surface*, Approximation Theory (Z. Ciesielski and J. Musielak, eds.), Reidel, Dordrecht, 1975, pp. 63–70.
5. —, *Approximation uniforme qualitative sur des ensembles non bornés*, Les Presses de l'Université de Montréal, 1982.
6. M. Nakai and L. Sario, *Classification theory of Riemann surfaces*, Springer-Verlag, 1970.
7. R. Narasimhan, *Analysis on real and complex manifolds*, North-Holland, 1968.