ON THE MARCHAUD-TYPE INEQUALITY

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ABSTRACT. Marchaud's inequality, which is valid in many function spaces, was strengthened by M. F. Timan [3] for L_p , $1 < p < \infty$, following a technique of A. Zygmund [5]. In both the above-mentioned articles the powerful Littlewood-Paley theorem is used. In the present paper a direct and, I believe much simpler, proof is given for that stronger Marchaud-type inequality. Moreover, the result will apply to a more general class of function spaces. It will be shown that it is sufficient that the "modulus of smoothness" of the norm is of "power-type" p and that translation is an isometry.

Let B be a Banach space of functions on $D \subset R^n$ on $D \subset T^n$ (where T is $[-\pi, \pi]$ and $-\pi$ is identified with π). We define the r modulus of smoothness of f by

$$(1) \quad \omega_r(f, t)_B = \sup_{|h| \leq t} \|\Delta_h^r f\|_B,$$

where

$$(2) \quad \Delta_h^r f(x) = \begin{cases} \sum_{k=0}^r \binom{r}{k} (-1)^{r-k} f(x + kh) & \text{if conv hull } (x, x + rh) \subset D, \\ 0 & \text{otherwise.} \end{cases}$$

The extension of the Marchaud inequality is given in the following theorem.

THEOREM 1. *Suppose B is a Banach space of functions on D satisfying*

- (a) $\|f(\cdot + h)\|_B = \|f(\cdot)\|_B$ for all $h \in R^n$, where D is either R^n or T^n , and
- (b) $(\|f + g\|_B + \|f - g\|_B)/2 \leq (\|f\|_B^p + M\|g\|_B^p)^{1/p}$ for all $f, g \in B$ and some fixed $p \geq 1$.

Then we have

$$(3) \quad \omega_r(f, t)_B \leq Ct^r \left\{ \int_t^A \frac{\omega_{r+1}(f, u)_B^p}{u^{rp+1}} du + \|f\|_B^p \right\}^{1/p}.$$

PROOF. Define $Tf \equiv T(h)f \equiv f(x + h)$ and observe that

$$(4) \quad T - I = \frac{1}{2}(T^2 - I) - \frac{1}{2}(T - I)^2 \quad \text{and} \quad T(T - I) = \frac{1}{2}(T^2 - I) + \frac{1}{2}(T - I)^2.$$

Using (4), (a) and (b), we now obtain

$$\begin{aligned} \|(T - I)^r f\| &= \frac{1}{2}\|(T - I)^r f\| + \frac{1}{2}\|T(T - I)^r f\| \\ &\leq \left(\frac{1}{2}\|(T - I)^{r-1}(T^2 - I)f\|^p + M\left\|\frac{1}{2}(T - I)^{r+1}f\right\|^p\right)^{1/p}. \end{aligned}$$

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We now estimate $\|2^{-i}(T-I)^{r-i}(T^2-I)^i f\|$ for $1 \leq i < r$ in the same way to derive

$$\begin{aligned} & \|2^{-i}(T-I)^{r-i}(T^2-I)^i f\| \\ & \leq (\|2^{-i-1}(T-I)^{r-i-1}(T^2-I)^{i+1} f\|^p \\ & \quad + M\|2^{-i-1}(T-I)^{r-i+1}(T^2-I)^i f\|_p^p)^{1/p}. \end{aligned}$$

Therefore, iterating r times, we have

$$\|(T-I)^r f\| \leq \left(\|2^{-r}(T^2-I)^r f\|^p + M \sum_{j=0}^{r-1} \|2^{-j-1}(T-I)^{r-j+1}(T^2-I)^j f\|^p \right)^{1/p}.$$

Recalling $\|(T+I)g\| \leq 2\|g\|$, we have

$$\|(T-I)^{r-j+1}(T^2-I)^j f\| \leq 2^j \|(T-I)^{r+1} f\|$$

and, therefore, for $M_1 = M \cdot r \cdot 2^{-p}$

$$(5) \quad \|(T-I)^r f\| \leq (\|2^{-r}(T^2-I)^r f\|^p + M_1 \|(T-I)^{r+1} f\|^p)^{1/p}.$$

As $T^2 f = T(2h)f$, we may use the same estimate for the first term on the right and repeat m times to obtain

$$\|(T-I)^r f\| \leq \left(2^{-m r p} \|(T^{2^m} - I)^r f\|^p + M_1 \sum_{k=0}^{m-1} 2^{-k r p} \|(T^{2^k} - I)^{r+1} f\|^p \right)^{1/p}.$$

Therefore,

$$\begin{aligned} \omega_r(f, t) & \equiv \sup_{|h| \leq t} \|(T(h) - I)^r f\| \\ & \leq \left(2^{-(m-1)r p} \|f\|^p + M_1 \sum_{k=0}^{m-1} 2^{-k r p} \omega_{r+1}(f, 2^k t)^p \right)^{1/p}. \end{aligned}$$

Choosing m such that $2^{m-1}t < A \leq 2^m t$, and using the monotonicity of $\omega_{r+1}(f, t)$ and that of $u^{-r p}$, we have, following the standard Marchaud-type calculations,

$$\omega_r(f, t) \leq C t^r \left(\|f\|^p + \int_t^A \frac{\omega_{r+1}(f, u)^p}{u^{r p + 1}} du \right)^{1/p},$$

which is what we had to prove.

Proving the main result of this paper, we will now make a few remarks which will connect the present result with the literature and extend it further.

REMARK A. In the literature of the geometry of Banach spaces (see Lindenstrauss and Tzafriri [1, Vol. II, 59]) the concept “modulus of smoothness” of the norm is given by

$$(6) \quad \rho_B(\tau) = \sup_{\substack{\|f\|=1 \\ \|g\|=\tau}} \left(\frac{1}{2} \|f+g\| + \frac{1}{2} \|f-g\| - 1 \right).$$

If $\rho_B(\tau) \leq K \tau^p$ the space is said to have “a modulus of smoothness of power type p ” [1, Vol. II, 61]. As $\rho_B(\tau)$ was calculated for some spaces, it is useful to note that

(b) i.e.,

$$(b) \quad (\|f+g\| + \|f-g\|)/2 \leq (\|f\|^p + M\|g\|^p)^{1/p}$$

is equivalent to $\rho_B(\tau) \leq K\tau^p$. If B satisfies (b), we have for $\|f\| = 1$ and $\|g\| = \tau$

$$\rho_B(\tau) \leq (1 + M\tau^p)^{1/p} - 1 \leq \frac{M\tau^p/p}{(1 + \xi)^{1-1/p}} \leq \frac{M\tau^p}{p} \quad (0 < \xi < M\tau^p).$$

To show the opposite implication, we now observe that for some M , $K \min(\tau^p, \tau) + 1 \leq (1 + M\tau^p)^{1/p}$. This follows for $\tau \leq 1$ from $K\tau^p \leq M\tau^p/p(1 + M)^{1-1/p} \leq (1 + M\tau^p)^{1/p} - 1$ with the choice (of M) such that $(1 + M)^{1/p} \geq 2pK$ (as $M \geq 1$), and for $\tau \geq 1$ from $K\tau \leq (M\tau^p)^{1/p} - 1 \leq (1 + M\tau^p)^{1/p} - 1$ with the choice $M^{1/p} \geq K + 1$. Using the above, we can write

$$\frac{1}{2}\|f + g\| + \frac{1}{2}\|f - g\| \leq \|f\| \left(K \min \left(\frac{\|g\|^p}{\|f\|^p}, \frac{\|g\|}{\|f\|} \right) + 1 \right) \leq \|f\| \left(1 + M \frac{\|g\|^p}{\|f\|^p} \right)^{1/p}.$$

REMARK B. As $\rho_{L_p}(\tau) \equiv \rho_p(\tau)$ was calculated to be

$$\rho_p(\tau) \leq K\tau^p \text{ for } 1 \leq p \leq 2 \quad \text{and} \quad \rho_p(\tau) \leq K\tau^2 \text{ for } 2 \leq p < \infty$$

for some $K \equiv K(p)$ ($K \geq 1$) (see [1, Vol. II, 63]), we have M. F. Timan's result [3] (see also A. F. Timan [4, 108]); that is,

$$\omega_r(f, t)_B \leq M \left(\|f\|_B^p + t^r \int_t^A \frac{\omega_{r+1}(f, t)_B^p}{u^{rp+1}} du \right)^{1/p},$$

where $p = p_1$ when $B = L_{p_1}$ for $1 \leq p_1 \leq 2$, $p = 2$ when $B = L_{p_1}$ for $2 \leq p_1 < \infty$ and $p = 1$ when $B = L_\infty$. A. Zygmund [5] showed that the above are best possible. One can show (b) directly using Clarkson's inequality for $1 < p_1 \leq 2$ and direct computations (about a page) for $2 \leq p_1 < \infty$.

REMARK C. There are other spaces which satisfy $\rho_B(\tau) \leq K\tau^p$. As renorming is a common tool in the geometry of Banach space, we observe that if a space has an equivalent norm $\|\cdot\|_1$ which is translation invariant as well as satisfying (b) for some p , then we may apply the theorem with $\|\cdot\|_1$ and after proving the estimate (3) with the norm $\|\cdot\|_1$ use the equivalence to show that (3) is satisfied with the original norm (whether it was translation invariant or not). If, however, (b) is shown to be satisfied for a norm which is not translation invariant, our proof falters, and I suspect the result is no longer valid (that is, not in all cases).

REMARK D. Theorem 1 holds for $L_p(D)$ with other domains D . In particular if D is open with minimally smooth boundary, a concept which has several equivalent descriptions (see Sharpley [2]), the theorem reads as follows:

THEOREM 2. Suppose D is open, $D \subset R^n$ and D has minimally smooth boundary, and set $\|f\|_{L_{p_1}(D)} = \|f\|_{p_1}$ and $\omega_m(f, t)_{L_{p_1}(D)} \equiv \omega_m(f, t)_{p_1}$. Then

$$(7) \quad \omega_r(f, t)_{p_1} \leq Ct^r \left(\|f\|_{p_1}^p + \int_t^A \frac{\omega_{r+1}(f, u)_{p_1}^p}{u^{rp+1}} du \right)^{1/p},$$

$$p = \begin{cases} p_1, & 1 \leq p_1 \leq 2, \\ 2, & 2 \leq p_1 < \infty, \\ 1, & p_1 = \infty. \end{cases}$$

PROOF. We follow Sharpley [2] to have an extension of f, Ef in R^n that satisfies

$$\omega_m(f, t)_{L_{p_1}(\Omega)} \leq \omega_m(Ef, t)_{L_{p_1}(R^n)} \leq c\omega_m(f, t)_{L_{p_1}(\Omega)},$$

which is valid for $m = r$ and $m = r + 1$ simultaneously. We use Theorem 1 on Ef and the above estimate to obtain (7).

REMARK E. The analogue for L_p , $0 < p < 1$, is very simple and probably known. As I could not find it in print, I am presenting here a somewhat more general theorem.

THEOREM 3. Suppose U is a linear space of functions on $D \subset R^n$, where $x \in D$ implies $x + h \in D$ for $h = (h_1, \dots, h_n)$, $h_i \geq 0$, and suppose that the quasinorm $= \|f\|$ on U satisfies: (I) $\|\alpha f\| = |\alpha| \|f\|$, (II) $\|f + g\|^p \leq (\|f\|^p + M\|g\|^p)$ for some fixed M and p where $0 < p < 1$, and (III) $\|T(h)f\| \leq \|f\|$ for $T(h)f(x) = f(x + h)$ and $h_i \geq 0$. Then we have

$$(8) \quad \omega_r(f, t) \leq Ct^r \left(\|f\|^p + \int_t^A \frac{\omega_{r+1}(f, u)^p}{u^{rp+1}} du \right)^{1/p},$$

where

$$\omega_m(f, t) \equiv \sup_{\substack{|h| \leq t \\ h_i \geq 0}} \|\Delta_h^m f\| \equiv \sup_{\substack{|h| \leq t \\ h_i \geq 0}} \|(T(h) - I)^m f\|.$$

PROOF. Using $T - I = \frac{1}{2}(T^2 - I) - \frac{1}{2}(T - I)^2$, $T = T(h)$, we have

$$\begin{aligned} \|(T - I)^r f\|^p &\leq \left(\frac{1}{2}\right)^p \|(T^2 - I)(T - I)^{r-1} f\|^p + M\left(\frac{1}{2}\right)^p \|(T - I)^{r+1} f\|^p \\ &\leq \left(\frac{1}{2}\right)^{rp} \|(T^2 - I)^r f\|^p + M_1 \|(T - I)^{r+1} f\|^p \end{aligned}$$

as $\|(T^2 - I)^i (T - I)^{r-i+1} f\|^p \leq (1 + M)^i \|(T - I)^{r+1} f\|^p$ and

$$\sum_{j=0}^{r-1} (2^{-j-1})^p (1 + M)^j \leq M_2.$$

Repeating the process and using the monotonicity of $\omega_{r+1}(f, t)_p$ and that of u^{-rp} , we obtain the result in the standard way.

The quasinorm $(\int_D |f|^p)^{1/p}$ for $0 < p \leq 1$ satisfies (I), (II), and (III) of Theorem 3 with $M = 1$. The example, $f(x) = |x|^{p/(p-1)} \psi(x)$ where

$$\psi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2, \end{cases} \quad \psi(x) \in C^\infty(R),$$

shows that the result is best possible for $L_p(R)$ and $r = 1$. This is the same example used by Zygmund to show that, for $r = 1$, $1 < p \leq 2$, and $L_p(T)$, (7) is best possible.

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