THE OPERATOR $i d/dx$, ON $C[0, 1]$, IS C^1 -SCALAR

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ABSTRACT. We consider closed, unbounded linear operators A , on a Banach space, X , with discrete real spectrum, with corresponding eigenvectors, $\{x_k\}_{k=-\infty}^{\infty}$, such that $Ax_k = a_k x_k$, $a_k \leq a_{k+1}$, for all k , and $\lim_{k \rightarrow \pm\infty} |a_k| = \infty$. For arbitrary n , we present necessary and sufficient conditions for A to be C^n -scalar.

Letting $F(s)$ be the projection whose range is the $\text{span}\{x_k | a_k \text{ is between } 0 \text{ and } s\}$, null-space is the closure of $\text{span}\{x_k | a_k \text{ is not between } 0 \text{ and } s\}$, we show that A is C^1 -scalar if and only if the series

$$\sum_{a_k > 0} \varphi(F(a_k)x) \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) \quad \text{and} \quad \sum_{a_k < 0} \varphi(F(a_k)x) \left(\frac{1}{a_{k-1}} - \frac{1}{a_k} \right)$$

both converge absolutely, for all φ in X^* , x in X . As a corollary, we get that $i d/dx$, on $\{f \text{ in } C[0, 1] | f(0) = f(1)\}$, is C^1 -scalar. Also, $i d/dx$, on $L^1[0, 1]$, is C^1 -scalar.

Introduction. We consider closed linear operators, A , with eigenvectors $\{x_k\}_{k=-\infty}^{\infty}$, such that $Ax_k = a_k x_k$, and $D \equiv \text{span}\{x_k\}$ is dense (see Standing Hypotheses 4). Here, the functional calculus, $f \rightarrow f(A)$, is algebraically clear, at least on D ,

$$f(A) \left(\sum_{k=-N}^N c_k x_k \right) \equiv \sum_{k=-N}^N f(a_k) c_k x_k.$$

If $f(A)$ is bounded, it has a unique extension to the entire space. Thus the *continuity* of the map $f \rightarrow f(A)$ becomes essential to its construction.

A bounded operator is C^n -scalar if it has a functional calculus for functions with n continuous derivatives (see Definitions 2 and 3). An operator on a reflexive Banach space is C^0 -scalar if and only if it's a scalar-type spectral operator (see [4, Theorem 6.10]). A thorough analysis of C^n -scalar operators appears in [5]. An operator with real spectrum is *well bounded* if it has a functional calculus for absolutely continuous functions. See [4, Part 5], for basic material on well-bounded operators. A more general functional calculus appears in [2]. These concepts are extended to unbounded operators in [7].

It is well known that the operator $i d/dx$, on $L^p[0, 1]$, is a scalar-type spectral operator if and only if $p = 2$. When $1 < p < \infty$, the operator is well-bounded (see [1, (4.47)]). These facts are equivalent to the fact that, when $1 < p < \infty$, the Fourier series of any function, f , in L^p , converges to f in L^p , with unconditional convergence guaranteed only when $p = 2$.

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When $p = 1$ or ∞ , the operator $i d/dx$, on $L^p[0, 1]$, fails to be well bounded. Hence, to consider questions of uniform or L^1 convergence, we need to consider a weaker functional calculus.

We show that $i d/dx$, on $C[0, 1]$ or $L^1[0, 1]$, is C^1 -scalar (Corollaries 15, 16). Our most general result is Theorem 11, giving, for arbitrary n , necessary and sufficient conditions for an operator satisfying Standing Hypotheses 4, as in the Abstract, to be C^n -scalar. For $n = 1$, Corollary 13 gives a more workable characterization, which is used to prove Corollaries 15 and 16.

We are indebted to William Bade for introducing us to the subject of the title of our paper.

All operators are linear, on a Banach space, usually labelled X . X^* will be the adjoint of X , and $B(X)$ will be the space of all bounded linear operators from X to itself.

DEFINITION 1. We denote by $C^n[a, b]$, the set of all complex-valued functions on $[a, b]$, with n continuous derivatives.

$$\|f\|_{C^n[a, b]} \equiv \max_{0 \leq k \leq n} \|f^{(k)}\|_{\infty}, \quad \text{where } \|g\|_{\infty} \equiv \max\{|g(t)| \mid t \text{ is in } [a, b]\}.$$

If 0 is in $[a, b]$, then $C_0^n[a, b] \equiv \{f \text{ in } C^n[a, b] \mid f^{(k)}(0) = 0, \text{ when } k < n\}$.

DEFINITION 2. The bounded operator, A , is C^n -scalar on $[a, b]$ if it has a functional calculus for $C^n[a, b]$, that is, there exists a continuous algebra homomorphism, $f \rightarrow f(A)$, from $C^n[a, b]$ into $B(X)$, taking $f_0(t) \equiv 1$ to I , the identity operator, and $f_1(t) \equiv t$ to A .

DEFINITION 3. The closed (possibly unbounded) operator, A , with real resolvent, is C^n -scalar if the bounded operator $(r - A)^{-1}$ is C^n -scalar, whenever r is real and in the resolvent set of A . It is sufficient to have one value of r , for which $(r - A)^{-1}$ is C^n -scalar. This definition is equivalent to Marschall's definition of an operator of class C^n ($\hat{C}$) (see [7], Definition 2.2(b), Definition 3.1 and Corollary 3.9), and generalizes Definition 2.

STANDING HYPOTHESES 4. Throughout this paper, A will be a closed operator with discrete real spectrum, with corresponding eigenvectors $\{x_k\}_{k=-\infty}^{\infty}$, $Ax_k = a_k x_k$, with $\lim_{k \rightarrow \pm\infty} |a_k| = \infty$, $a_k \leq a_{k+1}$, for all k , and $\text{span}\{x_k\}_{k=-\infty}^{\infty}$, dense.

DEFINITION 5. Let $D \equiv \text{span}\{x_k\}_{k=-\infty}^{\infty}$. For any function, f , on the real line, define the operator $f(A)$, on D , by

$$f(A) \left(\sum_{k=-N}^N c_k x_k \right) \equiv \sum_{k=-N}^N f(a_k) c_k x_k.$$

When r is in the resolvent of A , then $(r - A)^{-1}$, on D , is $g_r(A)$, where $g_r(t) \equiv (r - t)^{-1}$. Define the operator $f((r - A)^{-1})$ on D , by

$$f((r - A)^{-1}) \equiv (f \circ g_r)(A).$$

Note that, when $f(t) = t^k$, $f((r - A)^{-1}) = (r - A)^{-k}$.

DEFINITION 6. Let E be the atomic projection-valued measure defined by

$$E(\{s\}) = \text{projection with range equal to } \text{span}\{x_k \mid a_k = s\}, \\ \text{null-space equal to the closed span of } \{x_k \mid a_k \neq s\}.$$

(See Standing Hypotheses 4.)

For φ in X^* , x in X , $E_{\varphi,x}$ is the complex-valued measure defined by $E_{\varphi,x}(B) \equiv \varphi(E(B)x)$, for all Borel sets B .

Note that $E(B) = \chi_B(A)$, on D (see Definition 5), so that, for φ in X^* , x in D ,

$$\varphi(f(A)x) = \int_{-\infty}^{\infty} f(s) dE_{\varphi,x}(s).$$

REMARK 7. When $A = id/dx$, on $C[0, 2\pi]$, or $L^p[0, 2\pi]$, then $x_k(t) \equiv e^{-ikt}$ (see Standing Hypotheses 4), D is the set of all trigonometric polynomials (see Definition 5), and $[E(\{k\})](f) = \hat{f}(k)x_k$, for any integer k , where $\hat{f}(k)$ is the k th Fourier coefficient of f .

In order to state our most general result (Theorem 11), we need some more terminology.

DEFINITION 8. Suppose r is a real number in the resolvent of A , with $a_N < r < a_{N+1}$ (see Standing Hypotheses 4). For m a complex-valued Borel measure on the real line, define the absolutely continuous measure $(T_r m)$ by

$$\frac{d}{ds}(T_r m)(s) \equiv \begin{cases} \frac{m[a_{N+1}, s]}{(r-s)^2}, & s \geq a_{N+1}, \\ \frac{-m[s, a_N]}{(r-s)^2}, & s \leq a_N, \\ 0, & a_N < s < a_{N+1}. \end{cases}$$

LEMMA 9. If φ is in X^* , x is in D , and k is a nonnegative integer, then $(T_r^k E_{\varphi,x})$ is of bounded variation on the real line.

PROOF. Suppose $x = \sum_{k=-N}^N c_k x_k$ is in D . Let "var" be "total variation". Then, $\text{var}(E_{\varphi,x}) \leq \sum_{k=-N}^N |c_k| |\varphi(x_k)|$, which is finite.

A short calculation shows that, for any Borel measure m ,

$$\text{var}(T_r m) \leq (\text{var}(m))(1/(a_{N+1} - r) + 1/(r - a_N)), \quad \text{where } a_N < r < a_{N+1}.$$

Thus, the result follows by induction on k .

LEMMA 10. Suppose r is a real number in the resolvent of A and $1/b$ is the distance from r to the spectrum of A . Then, for any f in $C_0^n[-b, b]$, φ in X^* , x in D , we have

$$\varphi(f((r-A)^{-1})x) = (-1)^n \int_{|s-r| \geq 1/b} f^{(n)} \left(\frac{1}{r-s} \right) d(T_r^n E_{\varphi,x}(s)).$$

PROOF. There exists N so that $a_N < r < a_{N+1}$ (see Standing Hypotheses 4). For any $k < n$, using the fact that $\lim_{s \rightarrow \pm\infty} f^{(k)}(1/(r-s)) = 0$ (see Definition 1), and $T_r^k E_{\varphi,x}(a_N, a_{N+1}) = 0$ and Lemma 9, we have the following integration by

parts

$$\begin{aligned}
& \int_{|s-r| \geq 1/b} f^{(k)} \left(\frac{1}{r-s} \right) d(T_r^k E_{\varphi, x}(s)) \\
&= \left(\int_{-\infty}^{a_N} + \int_{a_{N+1}}^{\infty} \right) \left(f^{(k)} \left(\frac{1}{r-s} \right) d(T_r^k E_{\varphi, x}(s)) \right) \\
&= -f^{(k)} \left(\frac{1}{r-s} \right) \int_s^{a_N} d(T_r^k E_{\varphi, x}(s)) \Big|_{-\infty}^{a_N} \\
&\quad + \int_{-\infty}^{a_N} \frac{d}{ds} \left(f^{(k)} \left(\frac{1}{r-s} \right) \right) \int_s^{a_N} d(T_r^k E_{\varphi, x}(t)) ds \\
&\quad + f^{(k)} \left(\frac{1}{r-s} \right) \int_{a_{N+1}}^s d(T_r^k E_{\varphi, x}(t)) \Big|_{a_{N+1}}^{\infty} \\
&\quad - \int_{a_{N+1}}^{\infty} \frac{d}{ds} \left(f^{(k)} \left(\frac{1}{r-s} \right) \right) \int_{a_{N+1}}^s d(T_r^k E_{\varphi, x}(t)) ds \\
&= - \left(\int_{-\infty}^{a_N} + \int_{a_{N+1}}^{\infty} \right) \left(f^{(k+1)} \left(\frac{1}{r-s} \right) d(T_r^{k+1} E_{\varphi, x}(t)) \right) \\
&= - \int_{|s-r| \geq 1/b} f^{(k+1)} \left(\frac{1}{r-s} \right) d(T_r^{k+1} E_{\varphi, x}(t)).
\end{aligned}$$

Hence, integrating the formula

$$\varphi(f((r-A)^{-1})x) = \int_{|s-r| \geq 1/b} f \left(\frac{1}{r-s} \right) dE_{\varphi, x}(s)$$

(see comments after Definitions 5 and 6) by parts n times gives the results.

THEOREM 11. *The operator A is C^n -scalar if and only if, for all r in the resolvent of A , φ in X^* , x in X , the complex-valued measure $(T_r^n E_{\varphi, x})$ is of bounded variation on the real line (see Definitions 6 and 8).*

PROOF. Let r be a real number in the resolvent of A . Let $1/b$ be the distance from r to the spectrum of A (so that the spectrum of $(r-A)^{-1}$ is contained in $[-b, b]$).

Suppose A is C^n -scalar. Then $(r-A)^{-1}$ is C^n -scalar on $[-b, b]$ (see Definitions 3 and 2). This implies that, for any φ in X^* , x in X , there exists a unique signed measure $m_{\varphi, x}$ such that for all f in $C_0^n[-b, b]$ (see Definition 1),

$$\varphi(f((r-A)^{-1})x) = (-1)^n \int_{|s-r| > 1/b} f^{(n)} \left(\frac{1}{r-s} \right) dm_{\varphi, x}(s).$$

By Lemma 10, $m_{\varphi, x} = T_r^n E_{\varphi, x}$, whenever x is in D (see Definition 5).

Let G be an arbitrary bounded Borel subset of the real line. Define operators R and S by $\varphi(Rx) \equiv m_{\varphi, x}(G)$, $\varphi(Sx) \equiv (T_r^n E_{\varphi, x})(G)$, for φ in X^* , x in X .

By Lemma 10, $Sx = Rx$, for all x in D . Since G is bounded, S and R are bounded operators; since D is dense, $S = R$.

It follows that, for any φ in X^* , x in X , the total variation of $T_r^n E_{\varphi, x}$ equals the total variation of $m_{\varphi, x}$, which is finite, as desired.

Conversely, suppose $\int_{-\infty}^{\infty} d|T_r^n E_{\varphi, x}(s)|$ is finite, for any φ in X^* , x in X . For any polynomial p , the argument given in the proof of Lemma 10 shows that

$$\begin{aligned} \varphi(p((r-A)^{-1})x) &\equiv \sum_{k=0}^{n-1} \frac{p^{(k)}(0)}{k!} \varphi((r-A)^{-k}x) \\ &\quad + (-1)^n \int_{|s-r| \geq 1/b} p^{(n)}\left(\frac{1}{r-s}\right) d(T_r^n E_{\varphi, x}(s)), \end{aligned}$$

for φ in X^* , x in X .

We have

$$|\varphi(p((r-A)^{-1})x)| \leq \|p\|_{C^n} \left(\sum_{k=0}^{n-1} \frac{1}{k!} |\varphi((r-A)^{-k}x)| + \int_{-\infty}^{\infty} d|T_r^n E_{\varphi, x}(s)| \right),$$

so that, by applying the Banach-Steinhaus theorem to $\{p((r-A)^{-1})|p \text{ is a polynomial, } \|p\|_{C^n} \leq 1\}$, we obtain the following inequality, for some finite M :

$$\|p((r-A)^{-1})\| < M\|p\|_{C^n},$$

for all polynomials p . Since the polynomials are a dense subalgebra of $C^n[-b, b]$, this implies that the map $p \rightarrow p((r-A)^{-1})$ extends to a continuous algebra homomorphism, from $C^n[-b, b]$ into $B(X)$, as desired (see Definition 3).

DEFINITION 12. Define finite-dimensional projections $F(s)$ by

$$F(s) \equiv \begin{cases} E[0, s] & \text{if } s \geq 0, \\ -E[s, 0] & \text{if } s < 0 \end{cases}$$

(see Definition 6).

COROLLARY 13. *The operator A is C^1 -scalar if and only if, for any φ in X^* , x in X , the series*

$$\sum_{a_k > 0} \varphi(F(a_k)x) \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right)$$

and

$$\sum_{a_k < 0} \varphi(F(a_k)x) \left(\frac{1}{a_{k-1}} - \frac{1}{a_k} \right)$$

both converge absolutely.

PROOF. This can be seen to be Theorem 11, with $n = 1$, by the following calculation, with $a_N < r < 0 \leq a_{N+1}$.

$$\begin{aligned} \int_{-\infty}^{\infty} d|(T_r E_{\varphi, x})(s)| &= \int_{-\infty}^{a_N} |E_{\varphi, x}[s, a_N]| \frac{ds}{(r-s)^2} \\ &\quad + \int_{a_{N+1}}^{\infty} |E_{\varphi, x}[a_{N+1}, s]| \frac{ds}{(r-s)^2} \\ &= \left(\int_{-\infty}^{a_N} + \int_{a_{N+1}}^{\infty} \right) |\varphi(F(s)x)| \frac{ds}{(r-s)^2} \end{aligned}$$

(since $E(a_N, a_{N+1}) = 0$)

$$\begin{aligned} &= \sum_{k=-\infty}^{N-1} \int_{a_k}^{a_{k+1}} |\varphi(F(a_{k+1})x)| \frac{ds}{(r-s)^2} \\ &\quad + \sum_{k=N+1}^{\infty} \int_{a_k}^{a_{k+1}} |\varphi(F(a_k)x)| \frac{ds}{(r-s)^2} \\ &= \sum_{k=-\infty}^{N-1} |\varphi(F(a_{k+1})x)| \left(\frac{1}{r-a_k} - \frac{1}{r-a_{k+1}} \right) \\ &\quad + \sum_{k=N+1}^{\infty} |\varphi(F(a_k)x)| \left(\frac{1}{r-a_k} - \frac{1}{r-a_{k+1}} \right). \end{aligned}$$

By a comparison test, these series converge if and only if the series in the corollary converge absolutely.

COROLLARY 14. *If*

$$\sum_{a_k > 0} \|F(a_k)\| \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) \quad \text{and} \quad \sum_{a_k < 0} \|F(a_k)\| \left(\frac{1}{a_{k-1}} - \frac{1}{a_k} \right)$$

both converge, then A is C^1 -scalar.

COROLLARY 15. *Let A be $i d/dx$, on $X \equiv \{f \text{ in } C[0, 2\pi] | f(0) = f(2\pi)\}$, with domain of $A \equiv \{\text{differentiable } f \text{ in } X | f' \text{ is in } X\}$. Then A is C^1 -scalar.*

PROOF. For k an integer, let $x_k(t) \equiv e^{-ikt}$. We have $Ax_k = kx_k$.

Let $\hat{f}(k) \equiv \int_0^{2\pi} f(t)e^{ikt} dt / 2\pi$, the k th Fourier coefficient of f . For any f in X , we have

$$F(n)f = \begin{cases} \sum_{k=0}^n \hat{f}(k)x_k & \text{if } n \geq 0, \\ -\sum_{k=n}^{-1} \hat{f}(k)x_k & \text{if } n < 0. \end{cases}$$

Thus, if we define D_n by

$$D_n \equiv \begin{cases} \sum_{k=0}^n x_k, & n \geq 0, \\ -\sum_{k=n}^{-1} x_k, & n < 0, \end{cases}$$

we have $F(n)f = f * D_n$, the convolution of f with D_n , so that $\|F(n)\| \leq \|D_n\|_1$, which can be shown to be $O(\log |n|)$. (See [6, p. 47], for an almost identical kernel.)

Since $\sum_{n=1}^{\infty} (\log n)(1/n - 1/(n+1))$ converges, A is C^1 -scalar, by Corollary 14.

The same argument gives the following.

COROLLARY 16. *The operator $i d/dx$, on $L^1[0, 1]$, is C^1 -scalar.*

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