CESÀRO CONVERGENCE OF MARTINGALE DIFFERENCE SEQUENCES AND THE BANACH-SAKS AND SZLENK THEOREMS

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ABSTRACT. It is shown that, for $1 \leq p < +\infty$, any weakly null martingale difference sequence in $L^p[0, 1]$ is Cesàro convergent to zero in the L^p norm. This result combined with a theorem of Gaposhkin gives an easy proof of two theorems of Banach-Saks and Szlenk at once.

Introduction. We first recall the concept of martingale difference sequences. Let (Σ_n) be a nondecreasing sequence of sub- σ -algebras of Σ , the algebra of Lebesgue measurable sets in $[0, 1]$. If, for each n , f_n is a Σ_n -measurable function whose conditional expectation with respect to Σ_{n-1} satisfies $E(f_n, \Sigma_{n-1}) = 0$, we then say that the sequence (f_n) is a martingale difference sequence (m.d.s.) adapted to (Σ_n) . If every f_n is in $L^p = L^p[0, 1]$, we then say that (f_n) is an m.d.s. in L^p , where $1 \leq p < +\infty$.

Now we can state Gaposhkin's theorem (see [G]):

THEOREM A. *Let (g_n) be a weakly null sequence in L^p . Then there exists an m.d.s. (f_n) in L^p and a subsequence (h_n) of (g_n) such that the series $\sum_n \|h_n - f_n\|_p$ converges.*

PROOF. We outline Gaposhkin's proof for the sake of completeness. Let us suppose that $n_1 < \dots < n_{k-1}$ and $f_1, \dots, f_{k-1}$ have been defined so that $\|h_j - f_j\|_p < 2^{-j}$, and if $j \geq 2$, $E(f_j, \Sigma_{j-1}) = 0$, where Σ_j is the algebra generated by the simple functions $f_1, \dots, f_j$, and $h_j = g_{n_j}$, for $j = 1, \dots, k-1$. As $\lim_n \|E(g_n, \Sigma_{k-1})\|_p = 0$, we can choose $n_k > n_{k-1}$ with $\|E(h_k, \Sigma_{k-1})\|_p < 2^{-k-2}$, where $h_k = g_{n_k}$. We take a simple function f'_k so that $\|f'_k - h_k\|_p < 2^{-k-2}$. Then by setting $f_k = f'_k - E(f'_k, \Sigma_{k-1})$ we can go on with the induction.

Suppose now that we knew that (f_n) were Cesàro converging to zero in the L^p norm. Then as the series in Theorem A is converging, we would easily obtain from the following inequality that (h_n) was also Cesàro converging to zero:

$$\left\| n^{-1} \sum_{k=1}^n h_k \right\|_p \leq \left\| n^{-1} \sum_{k=1}^n (h_k - f_k) \right\|_p + \left\| n^{-1} \sum_{k=1}^n f_k \right\|_p.$$

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Thus we get the theorems proved by Banach and Saks in [BS] and by Szlenk in [S], asserting that any weakly null sequence in L^p admits a subsequence which is Cesàro converging in the norm, from this remark and the following

THEOREM. *If (f_n) is an m.d.s. in L^p which is weakly null in L^p , then its arithmetic means converge to zero in the L^p norm.*

PROOF. The proof in the case of $p > 1$ is essentially contained in Banach and Saks' paper [BS], and it is based on the inequality

$$|s + t|^p \leq |s|^p + p|s|^{p-1} \operatorname{sgn}(s)t + \sum_{k=2}^{[p]} \binom{p}{k} |s|^{p-k} |t|^k + A|t|^p$$

which holds for any real numbers s and t , and in which A is a finite constant and $[p]$ is the greatest integer less than or equal to p .

Using this inequality and assuming that (f_n) is an m.d.s. adapted to (Σ_n) with $\|f_n\|_p \leq 1$, we obtain $\|S_n\|_p \leq \|S_{n-1}\|_p + Bn^{p-2} + A$, where $S_n = \sum_{k=1}^n f_k$ and B is a finite constant, hence obtaining $\lim \|n^{-1}S_n\|_p = 0$, as shown by Banach and Saks.

We notice that

$$p \int |S_{n-1}(x)|^{p-1} \operatorname{sgn}(S_{n-1}(x)) f_n(x) dx$$

vanishes because $|S_{n-1}|^{p-1} \operatorname{sgn}(S_{n-1})$ is Σ_{n-1} -measurable and is in L^q , if $p^{-1} + q^{-1} = 1$.

Now we assume that $p = 1$ and that (f_n) is an equi-integrable m.d.s. in L^1 adapted to (Σ_n) with $\|f_n\|_1 \leq 1$ for every n .

Let us fix $\varepsilon > 0$, and let M be a finite constant to be fixed later. Let $g_n = f_n \chi_{(|f_n| \leq M)}$, where, as usual, $\chi_{(|f_n| \leq M)}$ is the characteristic function of the set $(|f_n| \leq M) = \{x: |f_n(x)| \leq M\}$.

Let $h_n = E(g_n, \Sigma_{n-1})$ for $n > 1$, and $h_1 = g_1$. Then h_n is Σ_{n-1} -measurable, $(g_n - h_n)$ is an m.d.s. adapted to (Σ_n) and $\|g_n - h_n\|_\infty \leq 2M$.

Let $S_n = \sum_{k=1}^n f_k$, $S'_n = \sum_{k=1}^n (f_k - g_k)$, $S''_n = \sum_{k=1}^n (g_k - h_k)$, and $S'''_n = \sum_{k=1}^n h_k$.

Then

$$\begin{aligned} \|S_n\|_1 &\leq \|S'_n\|_1 + \|S''_n\|_1 + \|S'''_n\|_1 \\ &\leq \sum_{k=1}^n \|f_k \chi_{(|f_k| > M)}\|_1 + \|S''_n\|_2 + \sum_{k=1}^n \|h_k\|_1. \end{aligned}$$

We have $\|S''_n\|_2^2 = \sum_{k=1}^n \|g_k - h_k\|_2^2 \leq 4nM^2$, because $(g_n - h_n)$ is an m.d.s. in L^2 .

As (f_n) is equi-integrable, we can choose $\delta > 0$ such that for any measurable A whose measure does not exceed δ , $\int_A |f_n(x)| dx \leq \varepsilon/3$ for every n . Then, we fix M with $M > \delta^{-1}$, obtaining $\|f_n \chi_{(|f_n| > M)}\|_1 \leq \varepsilon/3$ for every n .

Furthermore, let u be a Σ_{n-1} -measurable and essentially bounded function. Then,

$$\begin{aligned}\int h_n(x)u(x) dx &= \int f_n(x)u(x)\chi_{(|f_n|\leq M)}(x) dx \\ &= - \int f_n(x)u(x)\chi_{(|f_n|>M)}(x) dx.\end{aligned}$$

Thus $|\int h_n(x)u(x) dx| \leq \|u\|_\infty \|f_n\chi_{(|f_n|>M)}\|_1$, and so $\|h_n\|_1 \leq \varepsilon/3$, since h_n is Σ_{n-1} -measurable.

Putting all the parts together, we obtain

$$\|S_n\|_1 \leq n\varepsilon/3 + 2Mn^{1/2} + n\varepsilon/3,$$

finishing the proof.

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