AN APPLICATION OF BANACH LIMITS

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ABSTRACT. Let l_∞ denote the Banach space of bounded sequences, σ an injection of the set of positive integers into itself having no finite orbits, and T the operator defined on l_∞ by $Ty(n) = y(\sigma n)$. A positive linear functional $\mathcal{L}$ with $\|\mathcal{L}\| = 1$, is called a σ -mean if $\mathcal{L}(y) = \mathcal{L}(Ty)$ for all y in l_∞ . A sequence y is said to be σ -convergent, denoted $y \in V_\sigma$, if $\mathcal{L}(y)$ takes the same value, called σ -lim y , for all σ -means $\mathcal{L}$. P. Schaefer [6] gave necessary and sufficient conditions on a matrix A to ensure that $A(c) \subset V_\sigma$, where c is the space of convergent sequences, and additional conditions ensuring that σ -lim $Ay = \lim y$ for all $y \in c$, denoting the class of matrices satisfying these conditions by $(c, V_\sigma)_1$ and calling them the σ -regular matrices. In this paper, we use such matrices to find the sum of a sequence of Walsh functions.

1. Introduction. Let l_∞ and c be the Banach spaces of bounded and convergent sequences, respectively, with usual norm $\|y\| = \sup_k |y_k|$, and let BV be the Banach space of sequences of bounded variation normed by $\|y\| = \sum_k |y_k - y_{k-1}|$. We denote by V_σ , the set of bounded sequences all of whose invariant means are equal. In case σ is the translation mapping $n \rightarrow n + 1$, a σ -mean is often called a Banach limit and V_σ is the set of almost convergent sequences [3]. A σ -mean extends the limit functional on c in the sense that $\mathcal{L}(y) = \lim y$ for all $y \in c$ if and only if σ has no finite orbits (see [4 and 5]).

Schaefer [6] gave the following necessary and sufficient conditions for an infinite matrix $A = (a_{nk})$ to be σ -regular:

$$(1.1) \quad \|A\| = \sup_n \sum_k |a_{nk}| < +\infty,$$

$$(1.2) \quad a_{(k)} = \{a_{nk}\}_{n=1}^\infty \in V_\sigma \quad \text{with } \sigma\text{-limit zero for each } k,$$

and

$$(1.3) \quad a = \left\{ \sum_k a_{nk} \right\}_{n=1}^\infty \in V_\sigma \quad \text{with } \sigma\text{-limit } +1.$$

In this paper, we use such σ -regular matrices to sum the following sequence of Walsh functions. Let us define a sequence of functions $h_0(x), h_1(x), \dots, h_n(x)$ which satisfy the following conditions:

$$h_0(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1, \end{cases}$$

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$h_0(x+1) = h_0(x)$, and $h_n(x) = h_0(2^n x)$, $n = 1, 2, 3, \dots$. The functions $h_n(x)$ are called Rademacher's functions.

The Walsh functions are defined by

$$\begin{aligned}\phi_0(x) &= 1, \\ \phi_n(x) &= h_{n_1}(x)h_{n_2}(x) \cdots h_{n_r}(x), \quad 0 \leq x \leq 1,\end{aligned}$$

for $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, where the integers n_i are uniquely determined by $n_{i+1} < n_i$.

Let f be L -integrable and periodic with period 1, and let the Walsh-Fourier series of f be

$$\sum_{n=1}^{\infty} a_n \phi_n(x),$$

where

$$a_n = \int_0^1 f(x) \phi_n(x) dx$$

are called the Walsh-Fourier coefficients of f .

We recall some basic properties of Walsh functions (see [2]): For each fixed x and for all t ,

- (i) $\phi_n(x \dot{+} t) = \phi_n(x) \phi_n(t)$,
- (ii) $\int_0^1 f(x \dot{+} t) dt = \int_0^1 f(t) dt$, and
- (iii) $\int_0^1 f(t) \phi_n(x \dot{+} t) dt = \int_0^1 f(x \dot{+} t) \phi_n(t) dt$,

where $\dot{+}$ denotes the operation in the dyadic group, the set of all sequences $\bar{s} = \{s_n\}$, $s_n = 0, 1$ for $n = 1, 2, 3, \dots$, is addition modulo 2 in each coordinate.

Let

$$J_k(x) = \int_0^x \phi_k(t) dt, \quad k = 0, 1, 2, \dots$$

It is easy to see that $J_k(x) = 0$ for $x = 0, 1$.

2. The main result. We establish the following Theorem concerning σ -regular matrices.

THEOREM. Let z_x be the sequence defined by $z_x(k) = kJ_k(x)$. Let s_x be the sequence defined by $s_x(k) = ka_k\phi_k(x)$ for an L -integrable function f . Let A be a σ -regular matrix. Then the following statements about A are equivalent:

- (2.1) $A s_x \in V_\sigma$ with σ -limit zero, whenever x is a point at which $f(x) \in BV$.
- (2.2) $A z_x \in V_\sigma$ and has σ -limit zero for all x .

PROOF. We have

$$\begin{aligned}s_x(k) &= ka_k\phi_k(x) \\ &= k \int_0^1 f(t) \phi_k(t) \phi_k(x) dt \\ &= k \int_0^1 f(t) \phi_k(x \dot{+} t) dt \\ &= k \int_0^1 f(x \dot{+} t) \phi_k(t) dt, \\ &= k[f(x \dot{+} t) J_k(t)]_0^1 - \int_0^1 z_t(k) df(x \dot{+} t).\end{aligned}$$

Since $k[f(x+t)J_k(t)]_0^1 = 0$ by the fact that $J_k(1) = J_k(0) = 0$. Therefore, for every positive integer n ,

$$\sum_{k=1}^{\infty} \sum_{j=0}^p \frac{a(\sigma^j(n), k) s_x(k)}{p+1} = - \int_0^1 B_{p,n}(t) dh_x(t),$$

where

$$B_{p,n}(t) = \sum_{k=1}^{\infty} \sum_{j=0}^p \frac{a(\sigma^j(n), k)}{p+1} z_t(k)$$

and

$$h_x(t) = f(x+t).$$

Now, it is sufficient to show that $\int_0^1 B_{p,n}(t) dh_x(t) \rightarrow 0$, as $p \rightarrow \infty$, uniformly in n . Since, the Banach weak convergence theorem [1] states that $\int_0^1 g_n dh_x \rightarrow 0$ for all $h_x \in BV$ if and only if $\|g_n\| < M$ for all n and $g_n \rightarrow 0$ pointwise. Therefore, if we choose $g_n = \{B_{p,n}(t)\}$, i.e. the sequence of the A -transforms of the particular sequence $z = \{kJ_k(t)\}$ for any $t \in R$, and it says, in brief, that if A is a σ -regular matrix method, then $\|Az_t\| \leq \|A\|$, where $\|A\|$ is the norm of A regarded as a linear mapping from the Banach space c to the Banach space l_∞ . But this is immediate once it is observed that A also maps l_∞ to l_∞ , with the same norm, as is true of any method satisfying (1.1).

This completes the proof.

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