HOMOGENEOUS POLYNOMIALS ON THE BALL OF C^2

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ABSTRACT. Let B_2 be the open unit ball of C^2 . Let $T = \{(u, w) \in bB_2 : 1 - a \leq |u|^2 \leq a \text{ where } 1/2 < a < 1/2 + \sqrt{3}/4\}$. For each $n \in \mathbb{N}$ we construct a homogeneous polynomial $H_n: C^2 \rightarrow C^2$ of degree n such that $|H_n| \leq 1$ on B_2 and $|H_n| \geq c$ on T where $c > 0$ does not depend on n .

Aleksandrov [1] and Løw [4] proved recently that there are proper holomorphic maps from balls to higher dimensional polydiscs. In his paper Aleksandrov observed that this can also be done in a way similar to his proof [2] of the existence of inner functions on the ball B_N where he used Ryll-Wojtaszczyk polynomials ([6], see also [5]) except that one needs the following sharper version of the Ryll-Wojtaszczyk result:

THEOREM 0 [1]. *Given $N \in \mathbb{N}$ there are $M \in \mathbb{N}$, $c > 0$, and for each $n \in \mathbb{N}$ there are homogeneous polynomials P_{ni} , $1 \leq i \leq M$, on C^N of degree n , such that*

$$c \leq \sum_{i=1}^M |P_{ni}(z)|^2 \leq 1 \quad (z \in bB_N).$$

Aleksandrov pointed out that there is certain interest in obtaining estimates for the smallest number $M = M(N)$ for which the assertion of Theorem 0 holds and gave the estimate $N \leq M(N) \leq 25^{N-1} + 1$.

In the present note we consider only the case $N = 2$. By an explicit construction of polynomials we show that "locally" we have $M(2) = 2$:

THEOREM 1. *Let $1/2 < a < 1/2 + \sqrt{3}/4$ and let*

$$T = \{(u, w) \in bB_2 : 1 - a \leq |u|^2 \leq a\}.$$

There is $c > 0$ and for each $n \in \mathbb{N}$ there are homogeneous polynomials P_n, Q_n on C^2 , of degree n , such that

- (i) $|P_n(z)|^2 + |Q_n(z)|^2 \leq 1$ ($z \in bB_2$),
- (ii) $|P_n(z)|^2 + |Q_n(z)|^2 \geq c$ ($z \in T$).

Note that this implies that $M(2) \leq 4$: If U is the unitary map on C^2 given by

$$U = \begin{bmatrix} 2^{-1/2} & 2^{-1/2} \\ -2^{-1/2} & 2^{-1/2} \end{bmatrix}$$

then it is easy to see that $U(T) \cup T = bB_2$ provided that a is close to $1/2 + \sqrt{3}/4$ so putting $R_n = P_n \circ U$, $S_n = Q_n \circ U$ we get the following corollary.

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COROLLARY. *There is $c > 0$ and for each $n \in \mathbb{N}$ there are homogeneous polynomials P_n, Q_n, R_n, S_n on C^2 of degree n such that*

$$c \leq |P_n(z)|^2 + |Q_n(z)|^2 + |R_n(z)|^2 + |S_n(z)|^2 \leq 1 \quad (z \in bB_2).$$

REMARK. Granted the estimates in [1, §5] Corollary implies the existence of proper holomorphic maps from B_2 to the polydisc in C^{12} which does not seem to follow directly from [1 or 4].

Write $S = bB_2$. For each $n \in \mathbb{N}$, $n \geq 2$, and each j , $1 \leq j \leq n-1$, let $F_{nj}(u, w) = d_{nj}u^jw^{n-j}$ where $d_{nj} > 0$ is chosen so that $\sup |F_{nj}|$ over S equals 1. The absolute value of F_{nj} is $d_{nj}|u|^j|w|^{n-j} = d_{nj}t^{j/2}(1-t)^{n/2-j/2} = f_{nj}(t)$, i.e. a function of $t = |u|^2$ on $[0, 1]$. The peak points j/n of f_{nj} are evenly distributed on $(0, 1)$ so after reading the proof of Lemma 2 in the paper of Hakim-Sibony [3] one would hope that if $k(n) \in \mathbb{N}$ are chosen properly and if $g_{nj} = f_{nj}^{k(n)}$ then

$$(i) \quad \sum_{j=1}^{n-1} g_{nj}(t) \leq L \quad (0 \leq t \leq 1),$$

$$(ii) \quad g_{nj} - \sum_{\substack{k=1 \\ k \text{ even}, k \neq j}}^{n-1} g_{nk} \geq \delta \quad \text{on } [j/n - 1/2n, j/n + 1/2n]$$

if j is even and if j/n stays away from 0 and 1, and an analogous statement should hold for odd j where $L < \infty$ and $\delta > 0$ can be chosen independently of n . As we shall see, this is indeed the case and a more general version of this proves Theorem 1.

LEMMA. *Let $1/2 < a < 1/2 + \sqrt{3}/4$ and let $\alpha > 0$, $j \in \mathbb{N}$, $1 \leq j < \alpha$. Put*

$$\varphi_{\alpha j}(t) = (\alpha/j)^j (\alpha/(\alpha-j))^{\alpha-j} t^j (1-t)^{\alpha-j}.$$

Then

- (i) $\varphi_{\alpha j}$ increases on $(0, j/\alpha)$ and decreases on $(j/\alpha, 1)$,
- (ii) $\varphi_{\alpha j}(j/\alpha) = 1$,
- (iii) $\varphi_{\alpha j}(t) \leq \exp(-2\alpha(t - j/\alpha)^2)$ ($0 < t < 1$),
- (iv) given $\varepsilon > 0$ there is $\alpha_0 < \infty$ such that

$$\varphi_{\alpha j} \geq \exp \left(-\frac{1}{\alpha} \left(\frac{1}{8a(1-a)} + \varepsilon \right) \right) \quad \text{on } [j/\alpha - 1/2\alpha, j/\alpha + 1/2\alpha],$$

whenever $\alpha \geq \alpha_0$ and $1-a \leq j/\alpha \leq a$.

PROOF. (i) and (ii) are trivial. To prove (iii), write $\alpha - j = \beta$, $t = j/\alpha + \xi$ to get

$$\varphi_{\alpha j}(t) = f(\xi) = (\alpha/j)^j (\alpha/\beta)^\beta (j/\alpha + \xi)^j (\beta/\alpha - \xi)^\beta.$$

We want to prove that $-2\alpha\xi^2 \geq \log f(\xi)$ ($-j/\alpha < \xi < \beta/\alpha$). By (ii) this holds for $\xi = 0$ so it is enough to prove that

$$\begin{aligned} (\log f(\xi))' &\geq -4\alpha\xi \quad (-j/\alpha < \xi < 0), \\ &\leq -4\alpha\xi \quad (0 < \xi < \beta/\alpha), \end{aligned}$$

i.e.

$$\alpha/(j/\alpha + \xi)(\beta/\alpha - \xi) \geq 4\alpha \quad (-j/\alpha < \xi < \beta/\alpha)$$

which is obviously true.

To prove (iv), let $\varepsilon > 0$. By (i) we have to prove that $f(1/2\alpha)$, $f(-1/2\alpha)$ are both greater than or equal to

$$q(\alpha) = \exp\left(-\frac{1}{\alpha}\left(\frac{1}{8a(1-a)} + \varepsilon\right)\right)$$

whenever $\alpha \geq \alpha_0$, $1-a \leq j/\alpha \leq a$. Writing $j/\alpha = s$ we get

$$f(\pm 1/2\alpha) = (1 \mp 1/2s\alpha)^{s\alpha} (1 \pm 1/2(1-s)\alpha)^{(1-s)\alpha}$$

hence it is enough to prove that there is some $\alpha_0 < \infty$ such that

$$(1 - 1/2s\alpha)^{s\alpha} (1 + 1/2(1-s)\alpha)^{(1-s)\alpha} \geq q(\alpha) \quad (\alpha \geq \alpha_0)$$

whenever $1-a \leq s \leq a$, i.e.

$$\begin{aligned} g_s(\alpha) &= \frac{1}{\alpha^2} \left(\frac{1}{8a(1-a)} + \varepsilon \right) + s \log(1 - 1/2s\alpha) \\ &\quad + (1-s) \log(1 + 1/2(1-s)\alpha) \geq 0 \end{aligned}$$

whenever $\alpha \geq \alpha_0$, $1-a \leq s \leq a$. Since $\lim_{\alpha \rightarrow \infty} g_s(\alpha) = 0$ ($1-a \leq s \leq a$) it is enough to prove that $g'_s(\alpha) \leq 0$ ($\alpha \geq \alpha_0$, $1-a \leq s \leq a$), i.e.

$$-\frac{2}{\alpha} \left(\frac{1}{8a(1-a)} + \varepsilon \right) + \frac{1}{2-1/s\alpha} - \frac{1}{12+1/(1-s)\alpha} \leq 0,$$

i.e.

$$\frac{1}{2s(1-s)(2-1/s\alpha)(2+1/(1-s)\alpha)} \leq \frac{1}{8a(1-a)} + \varepsilon.$$

Since the term on the left does not exceed $1/(2a(1-a)(2-1/(1-a)\alpha)(2+1/(a\alpha)))$ when $1-a \leq s \leq a$, such an α_0 indeed exists. This completes the proof.

PROPOSITION. Let $p > 0$, $r > 0$, $e^{-pr^2} < 1/2$. Then

$$\sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} e^{-p(t-2jr)^2} \leq 4e^{-pr^2} \quad (-r \leq t \leq r).$$

PROOF. Let $-r \leq t \leq r$. for each j , $j \neq 0$,

$$e^{-p(t-2jr)^2} \leq e^{-p((2|j|-1)r)^2} \leq e^{-pj^2r^2}$$

so that the sum above does not exceed

$$2(e^{-pr^2} + e^{-p2^2r^2} + \dots) \leq 2(e^{-pr^2} + e^{-2pr^2} + \dots) \leq 4e^{-pr^2}$$

which completes the proof.

PROOF OF THEOREM 1. Observe first that if $n_0 \in \mathbf{N}$ it is enough to construct P_n, Q_n for $n \geq n_0$ as one can put $P_n(u, w) = Q_n(u, w) = u^n$, $1 \leq n \leq n_0$.

Let $a < b < 1/2 + \sqrt{3}/4$. We have $b(1-b) > 1/16$ which implies that there are δ , $0 < \delta < 1$, $\varepsilon > 0$ and $q \in \mathbf{N}$ such that

$$d = \exp\left(-\frac{q}{2}\left(\frac{1}{8b(1-b)} + \varepsilon\right)\right) - 4e^{-(q-\delta)} > 0.$$

By the lemma there is some $\alpha_0 < \infty$ such that

$$(1) \quad \varphi_{\alpha j} \geq \exp\left(-\frac{1}{\alpha}\left(\frac{1}{8b(1-b)} + \varepsilon\right)\right) \quad \text{on } [j/\alpha - 1/2\alpha, j/\alpha + 1/2\alpha]$$

whenever $\alpha \geq \alpha_0$ and $1 - b \leq j/\alpha \leq b$. We may assume that

$$(2) \quad \alpha_0 > 1/(b - a).$$

Suppose that $m_0 \in \mathbf{N}$ is so large that

$$(3) \quad m_0 > \alpha_0, \quad q(m_0/(m_0 + 1))^4 > q - \delta.$$

Let $n_0 \in \mathbf{N}$, $n_0 > qm_0^2$.

Let $n \in \mathbf{N}$, $n \geq n_0$. Choose $m \in \mathbf{N}$, $m \geq m_0$ such that

$$(4) \quad qm^2 \leq n < q(m + 1)^2.$$

Let k be the largest positive integer such that $kqm < n$. For each j , $1 \leq j \leq k$, define $P_{nj}(u, w) = c_{nj}u^{jqm}w^{n-jqm}$ where $c_{nj} > 0$ are chosen so that $\sup |P_{nj}|$ over S equals 1. Let $V = \{j: 1 \leq j \leq k, j \text{ even}\}$, $W = \{j: 1 \leq j \leq k, j \text{ odd}\}$ and define

$$P_n = \sum_{j \in V} P_{nj}, \quad Q_n = \sum_{j \in W} P_{nj}.$$

Clearly

$$|P_{nj}(u, w)| = c_{nj}t^{jqm/2}(1 - t)^{n/2 - jqm/2}$$

where $t = |u|^2$ and $(u, w) \in S$, so $|P_{nj}|^{2/qm} = \varphi_{\alpha j}$ where $\alpha = n/qm$. By Lemma (iii) we have

$$|P_{nj}| \leq e^{-n(t-j/\alpha)^2} \quad (0 \leq t \leq 1, 1 \leq j \leq k).$$

By (4), $(1/\alpha)^2 > m^2/(m+1)^4$ so by (4) and by Proposition with $r = 1/\alpha$ it follows that if $j_0 \in V$, $(u, w) \in S$, $j_0/\alpha - 1/\alpha \leq |u|^2 \leq j_0/\alpha + 1/\alpha$, then

$$\sum_{j \in V, j \neq j_0} |P_{nj}(u, w)| \leq 4e^{-n(1/\alpha)^2} \leq 4e^{-nm^2/(m+1)^4} \leq 4e^{-qm^2 \cdot m^2/(m+1)^4}$$

so by (3)

$$(5) \quad \sum_{j \in V, j \neq j_0} |P_{nj}(u, w)| \leq 4e^{-(q-\delta)} \\ (j_0 \in V, (u, w) \in S, j_0/\alpha - 1/\alpha \leq |u|^2 \leq j_0/\alpha + 1/\alpha)$$

and the same holds with V replaced by W .

By (3) and (4) $\alpha \geq \alpha_0$ so by (1) and (4)

$$(6) \quad |P_{nj}(u, w)| \geq \exp\left(-\frac{qm}{2} \cdot \frac{qm}{n} \left(\frac{1}{8b(1-b)} + \varepsilon\right)\right) \\ \geq \exp\left(-\frac{q}{2} \left(\frac{1}{8b(1-b)} + \varepsilon\right)\right)$$

if $1 \leq j \leq k$, $1 - b \leq j/\alpha \leq b$, $(u, w) \in S$ and $j_0/\alpha - 1/2\alpha \leq |u|^2 \leq j_0/\alpha + 1/2\alpha$. Now, by (5)

$$(7) \quad |P_n| \leq 1 + 4e^{-(q-\delta)}, \quad |Q_n| \leq 1 + 4e^{-(q-\delta)} \quad \text{on } S.$$

Further, if $j_0 \in V$, $1 - b \leq j_0/\alpha \leq b$ and if $(u, w) \in S$, $j_0/\alpha - 1/2\alpha \leq |u|^2 \leq j_0/\alpha + 1/2\alpha$ then by (5) and (6)

$$\begin{aligned} |P_n(u, w)| &\geq |P_{nj_0}(u, w)| - \sum_{j \in V, j \neq j_0} |P_{nj}(u, w)| \\ &\geq \exp\left(-\frac{q}{2}\left(\frac{1}{8b(1-b)} + \varepsilon\right)\right) - 4e^{-(q-\delta)} \\ &= d. \end{aligned}$$

Similarly, if $j_0 \in W$, $1 - b \leq j_0/\alpha \leq b$ and if $(u, w) \in S$, $j_0/\alpha - 1/2\alpha \leq |u|^2 \leq j_0/\alpha + 1/2\alpha$ then $|Q_n(u, w)| \geq d$. By (2) it follows that

$$|P_n(u, w)|^2 + |Q_n(u, w)|^2 \geq d^2 \quad ((u, w) \in T).$$

This, together with (7) and after replacing P_n, Q_n by $P_n/5, Q_n/5$, completes the proof of Theorem 1.

ADDED IN PROOF. The existence of proper holomorphic maps from B_2 to the polydisc in C^3 has been proved recently by Berit Stensønes (*Proper holomorphic mappings from strongly pseudoconvex domains in C^n to the unit polydisc in C^{n+1}* , Preprint).

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