WEAK (1, 1) BOUNDEDNESS OF SINGULAR INTEGRALS WITH NONSMOOTH KERNEL

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ABSTRACT. If $\Omega \in L^q(S^1)$ for some $q > 1$, $\int_{S^1} \Omega = 0$, and Ω is homogeneous of degree 0, then the operator defined in two dimensions by $T_\varepsilon f(x) = \int_{|y|>\varepsilon} f(x-y)\Omega(y)|y|^{-2} dy$ is of weak-type (1, 1) with bound independent of $\varepsilon > 0$.

Consider the two-dimensional operator defined on L^p , $1 \leq p < \infty$, by

$$T_\varepsilon f(x) \equiv \int_{|y|>\varepsilon} f(x-y)\Omega(y)|y|^{-2} dy$$

where Ω is homogeneous of degree zero, belongs to L^q on the unit circle for some $q > 1$, and has mean value zero. Note that no smoothness assumptions are placed upon Ω . It is well known [C-Z] that T_ε is of type (p, p) with bound independent of ε for $1 < p < \infty$. The proof is by the method of rotations. It has been an open problem, however, to determine the behavior of T_ε acting on L^1 . The main result of this paper is the following:

THEOREM. *Let Ω and T_ε be defined as above. Then T_ε is of weak-type (1, 1). More explicitly: $|\{ |T_\varepsilon f| > \alpha \}| \leq c \|\Omega\|_q \alpha^{-1} \|f\|_1$ where c is an absolute constant independent of ε .*

The proof of this theorem is based on ideas developed by Michael Christ in [C]. He proves that the related operator defined by

$$(*) \quad M_\Omega f(x) \equiv \sup_{r>0} \frac{1}{r^2} \int_{|y|<r} |f(x-y)| |\Omega(y)| dy$$

is weak (1, 1). What is new in this paper is the estimate for the cross terms (2), which do not arise in the analysis of (*). The proof of (2) uses an important observation contained in [C]; namely that a convolution of two kernels of the form $K = \Omega\psi$ (where $\psi \in C_0^\infty$ with annular support) can be expressed as the integral over $S^1 \times S^1$ of the convolution of two singular measures (3).

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PROOF OF THEOREM. We may assume $\Omega \in L^\infty(S^1)$. The general case follows by a limiting argument. We may also assume $\|\Omega\|_q = 1$. Fix $\alpha > 0$ and $f \in L^1$. Perform a standard Calderon-Zygmund decomposition to obtain nonoverlapping

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dyadic cubes $\{Q_n\}$ and functions g and b so that $f = g + b$, $\|g\|_2^2 \leq c\alpha\|f\|_1$, $b = \sum_n b_n$, $\text{supp } b_n \subseteq Q_n$, $\|b_n\|_1 \leq c\alpha|Q_n|$, b_n has mean value zero and $|\bigcup_n Q_n| \leq \alpha^{-1}\|f\|_1$.

Let $E \equiv \bigcup_n 5Q_n$, where $5Q_n$ has the same center as Q_n but side length 5 times that of Q_n . Then $|E| \leq C\alpha^{-1}\|f\|_1$.

As is well known, it is enough to show that

$$(1) \quad |\{x \in E^c : |T_\varepsilon b(x)| > \alpha\}| \leq (c/\alpha)\|f\|_1.$$

Define $B_i \equiv \sum b_n$, where the sum runs over those n for which $|Q_n| = 2^{2i}$. Let $\varphi \in C_0^\infty(\frac{1}{4}, 1)$, $\varphi \geq 0$ and $\sum_{j=-\infty}^\infty \varphi(x/2^j) \equiv 1$, for all $x > 0$. Let $K_0(x) \equiv (\Omega(x)/|x|^2)\varphi(|x|)$, and let $K_j(x) = 2^{-2j}K_0(2^{-j}x) = (\Omega(x)/|x|^2)\varphi(2^{-j}|x|)$. Note that

$$|T_\varepsilon b| \leq \left| \sum_i \sum_{j: 2^{j-2} > \varepsilon} B_i * K_j \right| + \sum_i \sum_{2^j \approx \varepsilon} |B_i| * |K_j|.$$

There are no more than three j terms appearing in the second sum, which is therefore bounded in L^1 norm by $C\|\Omega\|_1\|b\|_1$.

The first sum may be split into two parts, $T_{1,\varepsilon}b$ and $T_{2,\varepsilon}b$ where

$$T_{1,\varepsilon}b = \sum_i \sum_{\substack{j=i+2 \\ 2^{j-2} > \varepsilon}}^\infty B_i * K_j.$$

We need only prove (1) for $T_{1,\varepsilon}b$, because $\text{supp } T_{2,\varepsilon}b \subseteq E$. To see this, observe that for $j \leq i+1$ and $|Q_n| = 2^{2i}$, we have $\text{supp}(K_j * b_n) \subseteq 5Q_n$. Absolute convergence enables us to rearrange the summation defining $T_{1,\varepsilon}b$, so that

$$T_{1,\varepsilon}b = \sum_{s \geq 2} \sum_{j: 2^{j-2} > \varepsilon} B_{j-s} * K_j.$$

By the inequalities of Minkowski and then Tchebychev, it suffices to prove

$$\left\| \sum_{j: 2^{j-2} > \varepsilon} B_{j-s} * K_j \right\|_{L^r} \leq C2^{-\gamma s} s^\beta \alpha^{(r-1)/r} \|b\|_1^{1/r},$$

for some $\gamma, \beta > 0$, where $1/r = (1 + 1/q)/2$. As in [C], this follows by interpolation from the following.

LEMMA 1. For some $\eta > 0$,

$$\left\| \sum_j B_{j-s} * K_j \right\|_2^2 \leq C\alpha 2^{-\eta s} s \|\Omega\|_\infty^2 \|b\|_1.$$

PROOF OF LEMMA 1.

$$\begin{aligned} \left\| \sum_{j: 2^{j-2} > \varepsilon} B_{j-s} * K_j \right\|_2^2 &= \sum_j \|B_{j-s} * K_j\|_2^2 \\ &+ 2 \sum_j \langle B_{j-1-s} * K_{j-1}, B_{j-s} * K_j \rangle + 2 \sum_j \langle B_{j-2-s} * K_{j-2}, B_{j-s} * K_j \rangle \\ &+ 2 \sum_j \sum_{l \leq j-3} \langle B_{l-s} * K_l, B_{j-s} * K_j \rangle. \end{aligned}$$

By Lemma 2.1 of [C] and dilation invariance, the first sum on the right is bounded by $C2^{-\eta s}\alpha\|\Omega\|_\infty^2\sum_j\|B_{j-s}\|_1$. The next two sums can be handled in the same fashion after applying Schwarz's inequality. To estimate the last sum define $\tilde{K}_j(x) = K_j(-x)$ and note that

$$\begin{aligned} & \left| \sum_{j:2^{j-2}>\varepsilon} \sum_{\substack{l:2^{l-2}>\varepsilon \\ l\leq j-3}} \langle B_{l-s} * K_l, B_{j-s} * K_j \rangle \right| \\ &= \left| \sum_j \left\langle \left(\sum_l B_{l-s} * K_l * \tilde{K}_j \right), B_{j-s} \right\rangle \right| \\ &\leq \sum_{j=-\infty}^{\infty} \|B_{j-s}\|_1 \left\| \sum_{l\leq j-3} B_{l-s} * K_l * \tilde{K}_j \right\|_\infty. \end{aligned}$$

To prove the lemma, it is now enough to show that

$$(2) \quad \left\| \sum_{\substack{l\leq j-3 \\ 2^{l-2}>\varepsilon}} B_{l-s} * K_l * \tilde{K}_j \right\|_\infty \leq C\alpha 2^{-s}\|\Omega\|_\infty^2.$$

Fix x and j . Then there is an annulus $A_j(x)$ with $|A_j(x)| = C2^{2j}$ and $\text{supp } K_l * \tilde{K}_j(x - \cdot) \subseteq A_j(x)$, for all $l \leq j-3$. Thus, if $B_{l-s} = \sum_n b_n^{(l-s)}$ where $\text{supp } b_n^{(l-s)} \subseteq Q_n^{(l-s)}$ and $|Q_n^{(l-s)}| = 2^{2(l-s)}$, then

$$\left| \sum_{\substack{l\leq j-3 \\ 2^{l-2}>\varepsilon}} B_{l-s} * K_l * \tilde{K}_j(x) \right| \leq \sum_l \sum_n \left| \int b_n^{(l-s)}(y) K_l * \tilde{K}_j(x-y) dy \right|$$

with the sum running over those l and n for which

$$Q_n^{(l-s)} \cap A_j(x) \neq \emptyset.$$

Note that since $l \leq j-3$ we have $\sum_l \sum_n |Q_n^{(l-s)}| \leq C2^{2j}$.

To prove (2), we will use

LEMMA 2. *Fix s . Then for $l \leq j-3$, there is a decomposition $K_l * \tilde{K}_j = k_1^{l,j} + k_2^{l,j}$ such that*

- (i) $\|k_1^{l,j}\|_\infty \leq C\|\Omega\|_\infty^2 2^{-2j} 2^{-s}$,
- (ii) $\|\nabla k_2^{l,j}\|_\infty \leq C\|\Omega\|_\infty^2 2^{-2j} 2^{-l} s$,

where C is an absolute constant. Assume for now that Lemma 2 holds, so

$$\begin{aligned} & \sum_l \sum_n \int |b_n^{(l-s)}(y) k_1^{l,j}(x-y)| dy \\ &\leq C\|\Omega\|_\infty^2 2^{-s} 2^{-2j} \alpha \sum_l \sum_n |Q_n^{(l-s)}| \\ &\leq C\|\Omega\|_\infty^2 2^{-s} \alpha. \end{aligned}$$

Also, with $y' = \text{center of } Q_n(l-s)$, since b_n has mean value zero,

$$\begin{aligned} & \sum_l \sum_n \left| \int b_n^{(l-s)}(y) k_2^{l,j}(x-y) dy \right| \\ & \leq \sum_l \sum_n \int \left| b_n^{(l-s)}(y) \right| |y-y'| \|\nabla k_2^{l,j}\|_\infty dy \\ & \leq C \|\Omega\|_\infty^2 s 2^{-2j} \sum_l \sum_n 2^{-l} 2^{l-s} \|b_n^{l-s}\|_1 \\ & \leq C \|\Omega\|_\infty^2 2^{-s} s \alpha. \end{aligned}$$

Modulo Lemma 2, this concludes the proof of (2), and hence of Lemma 1 and of the Theorem.

PROOF OF LEMMA 2. Since $K_l * \tilde{K}_j(x) = 2^{-2j} K_{l-j} * \tilde{K}_0(2^{-j}x)$, it is enough to prove that for $m \geq 3$ and $\delta > 0$ given, there is a decomposition $k_{m,1} + k_{m,2} = K_{-m} * \tilde{K}_0$ such that

- (i) $\|k_{m,1}\|_\infty \leq C \|\Omega\|_\infty^2 \delta$,
- (ii) $\|\nabla k_{m,2}\|_\infty \leq C \|\Omega\|_\infty^2 2^m \log(\delta^{-1})$.

We will then take $m = j - l$ and $\delta = 2^{-s}$.

Set $\psi(t) = t^{-1} \varphi(t)$ so that $K_0(x) = \Omega(x) \varphi(|x|) |x|^{-2} = \Omega(x) \psi(|x|) |x|^{-1}$ and $K_{-m}(x) = 2^{2m} K_0(2^m x) = 2^m \Omega(x) \psi(2^m |x|) |x|^{-1}$.

As mentioned in the introduction, we have the representation

$$(3) \quad K_{-m} * \tilde{K}_0(x) = \iint_{S^1 \times S^1} u_\theta^m * \tilde{\mu}_\omega(x) d\theta d\omega$$

where μ_θ^m and $\tilde{\mu}_\omega$ are singular measures defined by

$$\begin{aligned} \langle \mu_\theta^m, h \rangle & \equiv \Omega(\theta) \int_0^\infty h(t\theta) 2^m \psi(2^m t) dt, \\ \langle \tilde{\mu}_\omega, h \rangle & \equiv \Omega(\omega) \int_0^\infty h(-s\omega) \psi(s) ds. \end{aligned}$$

The reader may verify (3) by using polar coordinates. Furthermore, by definition

$$\langle \mu_\theta^m * \tilde{\mu}_\omega, h \rangle = 2^m \Omega(\theta) \Omega(\omega) \int_0^\infty \int_0^\infty h(t\theta - s\omega) \psi(2^m t) \psi(s) dt ds.$$

By making the change of variable $x = t\theta - s\omega$ in the last equation we have

$$\langle \mu_\theta^m * \tilde{\mu}_\omega, h \rangle = \frac{2^m \Omega(\theta) \Omega(\omega)}{\Delta} \iint h(x_1, x_2) \psi\left(\frac{2^m \omega^* \cdot x}{\Delta}\right) \psi\left(\frac{\theta^* \cdot x}{\Delta}\right) dx_1 dx_2$$

with $\omega^* = (-\omega_2, \omega_1)$, $\theta^* = (-\theta_2, \theta_1)$ and

$$\Delta \equiv \begin{vmatrix} \theta_1 & -\omega_1 \\ \theta_2 & -\omega_2 \end{vmatrix} = |\omega^* \cdot \theta|.$$

Thus,

$$\mu_\theta^m * \tilde{\mu}_\omega(x) = 2^m \Omega(\theta) \Omega(\omega) \Delta^{-1} \psi\left(\frac{2^m \omega^* \cdot x}{\Delta}\right) \psi\left(\frac{\theta^* \cdot x}{\Delta}\right).$$

Note that

$$|\nabla_x \mu_\theta^m * \tilde{\mu}_\omega(x)| \leq (C 2^{2m} / \Delta^2) \|\Omega\|_\infty^2.$$

Define $\|\omega - \theta\| \equiv \min(|\omega - \theta|_{S^1}, |\omega + \theta|_{S^1})$, where $|(\cdot)|_{S^1}$ denotes arc length measured along the circle. We have

$$\Delta \equiv |\omega^* \cdot \theta| \approx \|\omega - \theta\|.$$

Also, because $|x| \approx C > 0$ for $x \in \text{supp } K_{-m} * \tilde{K}_0$ ($m \geq 3$), and since $\text{supp } \psi \subseteq (\frac{1}{4}, 1)$, $\mu_\theta^m * \tilde{u}_\omega(x)$ is nonvanishing only if

$$2^m \|\omega - x'\| \approx \Delta \approx \|\theta - x'\|$$

where $x' = x/|x|$.

Now set $K_{-m} * \tilde{K}_0 = k_{m,1} + k_{m,2}$ with

$$k_{m,1} \equiv 2^m \iint_{\|\theta - \omega\| \leq \delta} \frac{\Omega(\theta)\Omega(\omega)}{\Delta} \psi\left(\frac{2^m \omega^* \cdot x}{\Delta}\right) \psi\left(\frac{\theta^* \cdot x}{\Delta}\right) d\theta d\omega.$$

Then $k_{m,1}$ and $k_{m,2}$ satisfy the desired estimates.

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