TAME OBJECTS FOR FINITE COMMUTATIVE HOPF ALGEBRAS

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ABSTRACT. Let S be an A -module algebra for a commutative Hopf algebra A , both projective of the same rank over a commutative ring. Let $\mathbf{I}$ be the space of integrals in A . Then S is an invertible A -module iff it is a faithful module which satisfies the "trace surjectivity" condition that 1 is in $\mathbf{I}S$.

Let A be a commutative Hopf algebra, projective of rank n over a commutative ring R . Let S be a commutative algebra, also projective of rank n over R , and suppose we have made S an A -module algebra in the sense of Sweedler [4, p. 153]. That is, we have extended the R -module structure to an A -module structure that satisfies $b \cdot 1 = \varepsilon(b)1$ and $b \cdot st = \sum (b_i \cdot s)(c_i \cdot t)$ for the counit ε and the comultiplication $\Delta(b) = \sum b_i \otimes c_i$ in A . If A is also cocommutative, this means that we have an action of the finite commutative group scheme $\text{Spec}(A^*)$ on $\text{Spec}(S)$. Let $\mathbf{I}$ denote the set of integrals in A , so $\mathbf{I} = \{c \in A | bc = \varepsilon(b)c \text{ for all } b \in A\}$; this set is known to be a projective R -module of rank one [3, p. 592].

In a recent paper, Childs and Hurley [2] introduced some conditions on S weaker than those making it a Galois object for A ; they call it a *tame* object if (in addition to having the same rank as A) it satisfies

- (i) S is a faithful A -module,
- (ii) $R = \{s \in S | b \cdot s = \varepsilon(b)s \text{ for all } b \in A\}$, and
- (iii) $\mathbf{I} \cdot S = R$.

When R is an integral domain and certain supplementary conditions are satisfied, Childs and Hurley showed that any tame object is actually an invertible A -module. (Taking their terminology from the case where A is a commutative group algebra, they describe this by saying that S has "local normal bases", reflecting the fact that when R is local the ring S will then—like A^* —be free over A .) In this note, I shall show that the conclusion actually holds in full generality for arbitrary R . Furthermore, condition (ii) can be dropped, and (i) can be weakened; the "trace surjectivity" condition (iii) is the crucial property.

THEOREM. *Let R be a commutative ring, A a commutative Hopf algebra projective of rank n over R . Let $\mathbf{I}$ be the integrals in A , and let S be an A -module algebra also projective of rank n over R . The following are equivalent:*

- (1) S is tame in the sense of Childs and Hurley.
- (2) $S_Q \neq 0$ for all $Q \in \text{Spec}(A)$, and $1 \in \mathbf{I} \cdot S$.
- (3) S is an invertible A -module.

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PROOF. As S is finite over A , we know by [1, p. 133] that the primes Q with $S_Q \neq 0$ are those containing the annihilator of S ; thus (1) clearly implies (2).

Next we show that it suffices to prove (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) when R is local. Let P run through $\text{Spec}(R)$. We know in advance that S is finitely presented over A , since it is so over R , and hence by [1, p. 138] we see that condition (3) is true iff it is true over all localizations R_P . Similarly, as mentioned in [2, Corollary 2.6], each statement in the definition of tameness is true over R iff it is true for each R_P . Finally, $\mathbf{I}_P$ is the set of integrals in A_P , and the primes of A_P come from the primes of A containing P , so condition (2) implies the corresponding conditions over all R_P .

Assuming R local from now on, we show first that (3) $\Rightarrow$ (1). (Part of this is proved in [2, Theorem 2.7].) Since R is local, A is semilocal, and hence our invertible S is actually free over A , say $S = A \cdot t$ for some t . Similarly, $\mathbf{I}$ is a free R -module, say $\mathbf{I} = Rg$, and then $\mathbf{I} \cdot S$ is $R(g \cdot t)$. Obviously our S is a faithful A -module. We can write $1 = c \cdot t$ for some c in A , and we have $b \cdot (c \cdot t) = b \cdot 1 = \varepsilon(b)1 = \varepsilon(b)c \cdot t$ for all b . As S is free, this equation implies $b \cdot c = \varepsilon(b)c$. Thus c is in $\mathbf{I}$ and 1 is in $\mathbf{I} \cdot S$, so $R \subseteq \mathbf{I} \cdot S$. Similarly, any s in S that satisfies $b \cdot s = \varepsilon(b)s$ for all b lies in $\mathbf{I} \cdot S$, and thus we simply must show that $\mathbf{I} \cdot S$ is no bigger than R . But since S is projective rank n over R , it follows by [1, p. 176] that R is a direct summand of S as an R -module, say $S = R \oplus M$. Write $g \cdot t$ correspondingly as $r + m$. The element c above is in $\mathbf{I} = Rg$, say $c = r_1g$. Then $1 + 0 = 1 = r_1g \cdot t = r_1r + r_1m$, which tells us first that r_1 is invertible and then that $m = 0$. Thus indeed $\mathbf{I} \cdot S = Rg \cdot t = R$.

Finally, still with R local, we show that (2) $\Rightarrow$ (3). The first step is to show that S is a projective R -module. This is essentially proved in [2, Theorem 5.1], so we just briefly sketch the argument here. The multiplication map $A \otimes S \rightarrow S$ is A -linear. As R is local, $\mathbf{I}$ is free; let g be a basis element for it, with (say) $\Delta g = \sum g_i \otimes h_i$. By assumption there is an element $u \in S$ with $g \cdot u = 1$. Define a map $S \rightarrow A \otimes S$ by $x \mapsto \sum g_i \otimes u(s(h_i) \cdot x)$, where s denotes the antipode in A . It is easy to verify that $\sum g_i \cdot u(s(h_i) \cdot x) = x$, so this map is a section of $A \otimes S \rightarrow S$, and a slightly longer computation (using the fact that g is in $\mathbf{I}$) shows that the map is A -linear. Thus S is an A -module direct summand of $A \otimes S$, which itself is automatically projective over A since S is so over R .

To complete the proof, note that A is semilocal, and hence it can be decomposed as a finite product $A = Ae_1 \times \cdots \times Ae_r$, where the e_i are idempotents and Ae_i contains no nontrivial idempotents. Correspondingly then $S = e_1 \cdot S \times \cdots \times e_r \cdot S$, and $e_i \cdot S$ is projective (and finitely generated) over Ae_i . As Ae_i contains no nontrivial idempotents, each $e_i \cdot S$ has a well-defined rank m_i over Ae_i . Now each Ae_i is itself projective of some rank over the local ring R , and we have

$$n = \text{rank}_R S = \sum \text{rank}_R(e_i \cdot S) = \sum m_i[\text{rank}_R(Ae_i)].$$

But of course also

$$n = \text{rank}_R A = \sum \text{rank}_R(Ae_i).$$

Thus if any m_i is greater than one, some other m_j will have to be zero. But then we would have $S_Q = 0$ for the primes Q of A not containing e_j , and this contradicts our hypothesis. Hence all m_j are 1, and S has well-defined rank 1 over A . As A is semilocal, S is actually free. $\square$

COROLLARY. *Let $R \rightarrow T$ be any ring homomorphism. If S is tame for A over R , then $S \otimes T$ is tame for $A \otimes T$ over T .*

PROOF. This is not clear from the definition when $R \rightarrow T$, is not flat, but it is clear for the invertibility of the module. (The conditions in (2) also satisfy it.) $\square$

REMARK. If A is not commutative, conditions (i), (ii), and (iii) do not imply the existence of local normal bases, not even in the basic case where $R = \mathbf{C}$ and A is a group algebra $\mathbf{C}[G]$. Indeed, let G for example be the symmetric group on three elements. The element $\sum_G \sigma$ is of course a basis for the integrals. Let S be the algebra $\mathbf{C} \oplus M$, where M is a 5-dimensional $\mathbf{C}$ -space and $M^2 = 0$. To define a $\mathbf{C}[G]$ -module algebra structure on S , we simply need to have G acting on S as algebra automorphisms. Define the action by using a decomposition $M = M_1 \oplus M_2$, where M_1 is the 2-dimensional irreducible representation of G and M_2 is a 3-dimensional space carrying three copies of the signum representation $\sigma \cdot m = \text{sgn}(\sigma)m$. Clearly S does not carry the regular representation of $\mathbf{C}[G]$, and thus there is no normal basis. But it is trivial to check that this G -action preserves the algebra structure, and furthermore

(i) S is a faithful $\mathbf{C}[G]$ -module, as every type of irreducible representation occurs in it;

(ii) the s with $\sigma \cdot s = \varepsilon(\sigma)s$ for all σ are the fixed elements, which by construction are just the elements in $\mathbf{C}$; and

(iii) $\mathbf{I} \cdot S = [\sum \sigma] \cdot S = \mathbf{C}$.

It would be quite interesting to find criteria for the existence of local normal bases for noncommutative A , particularly in the important (group scheme) case where A is cocommutative.

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