THE CANCELLATION PROBLEM FOR FUNCTION FIELDS

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(Communicated by William C. Waterhouse)

ABSTRACT. Let L be a finitely generated extension of K . We call L rigid over K if the set of K -endomorphisms of L is finite. If L is rigid over K and x is transcendental over L , then L is invariant under automorphisms of $L(x)$ over K (Theorem 2). This result is used to show that the cancellation property holds for function fields of varieties of general type in characteristic 0.

Let L_1 and L_2 be finitely generated extensions of a field K and let x_i be transcendental over L_i , $i = 1, 2$. The Zariski problem [4] or what is more generally known as the cancellation problem asks: if $L_1(x_1) = L_2(x_2)$, must L_1 and L_2 be K -isomorphic? In general the answer is no. In [1], the authors construct a family of examples where L/K is of transcendence degree 2 and not rational, yet $L(x_1, x_2, x_3)$ is K -isomorphic to $K(x_4, x_5, x_6, x_7, x_8)$. The question remains whether this type of behavior is typical. An application of the main result of this paper shows it to be the exception. Specifically, it is shown that if L_1/K is the function field of a variety of general type of characteristic 0, then the cancellation property is true for L_1/K .

DEFINITION 1. L/K is called rigid if the set of K -endomorphisms of L is finite.

L is rigid over K if and only if the group of automorphisms of L over K is finite and L has no proper endomorphisms over K . This follows since once L/K has a proper endomorphism, repeated application will yield an infinite family of proper endomorphisms. For function fields of transcendence degree one over K , L is rigid over K if and only if the genus of L over K is at least two.

THEOREM 2. Let L be a finitely generated extension of K with x transcendental over L and let $\text{Aut}_K L(x)$ be the group of automorphisms of $L(x)$ which leave K fixed. If L/K is rigid and $\sigma \in \text{Aut}_K(L(x))$, then $\sigma(L) = L$.

PROOF. If L is algebraic over K , then L is the algebraic closure of K in $L(x)$ and hence $\sigma(L) \subseteq L$. Since L/K is rigid, $\sigma(L) = L$.

Thus we may assume L is not algebraic over K , and hence L is infinite. If $\sigma(L) \subseteq (L)$ then as above $\sigma(L) = L$. So assume we have a $\sigma \in \text{Aut}_K L(x)$ such that $\sigma(L) \not\subseteq (L)$ and we obtain a contradiction. Using σ we obtain a diagram

$$\begin{array}{ccc} L(x) & = & L^\sigma(x^\sigma) \\ & \swarrow \quad \searrow & \\ L & & L^\sigma \\ & \swarrow \quad \searrow & \\ & K & \end{array}$$

Received by the editors December 5, 1986 and, in revised form, April 7, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 12F20.

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0002-9939/88 \$1.00 + \$.25 per page

Let $a \in L$ and consider the $(x - a)$ -place of $L(x)$ whose residue field is L . Let L_a^σ be the residue field of L^σ under the restriction of the place and note we have $L \supseteq L_a^\sigma \supseteq K$. According to the ruled residue theorem [5], either L is algebraic over L_a^σ or L is ruled over the algebraic closure of L_a^σ in L . But if $L = L_1(z)$ with z transcendental over L_1 , then L is K -isomorphic to $L_1(z^2)$, which is not possible since L/K is rigid. Thus L must be algebraic over L_a^σ . Since the transcendence degree of L^σ/K has not dropped under the place, the place must give an isomorphism of L^σ into L . Once again since L/K is rigid, this isomorphism must be onto L . Thus each element $a \in L$ determines an isomorphism (over K), call it γ_a , of L^σ onto L . Thus each $\gamma_a \circ \sigma$ is an automorphism of L/K . Since L is rigid over K and infinite, some infinite subset of these, say $\gamma_{a_i} \circ \sigma$ where $i \in N$, all determine the same automorphism of L . Thus the γ_{a_i} must all define the same isomorphism of L^σ onto L . Let $\alpha \in L^\sigma \setminus L$. Then $\alpha = r(x)$ is a nonconstant rational function with coefficients in L . Thus $r(x) - r(a_1)$ is also a nonconstant rational function. Since $\gamma_{a_i}(r(x)) = \gamma_{a_1}(r(x))$, $r(x) - r(a_1)$ has an infinite number of distinct roots, a contradiction. Thus L is invariant under $\text{Aut}_K L(x)$.

COROLLARY 3. *Let L_i be finitely generated over K and x_i transcendental over L_i , $i = 1, 2$, and suppose $L_1(x_1) = L_2(x_2)$. If L_1/K is rigid, then $L_2 = L_1$.*

PROOF. Assume $L_2 \neq L_1$. Then $L_1 L_2$ properly contains both L_1 and L_2 and hence by Luroth's theorem $L_1 L_2$ is simple transcendental over both L_1 and L_2 . Thus we may assume $L_1 L_2 = L_2(x_2)$. Thus automorphisms of $L_2(x_2)/L_2$ are determined by where they map the elements of L_1 . Since L_1/K is rigid, L_1 is invariant under these automorphisms (Theorem 3). Thus the group of automorphisms of $L_2(x_2)$ over L_2 is finite. Thus L_2 is finite and both L_1 and L_2 are the algebraic closure of K in $L_2(x_2)$.

COROLLARY 4. *Let L_i be finitely generated over K and x_i transcendental over L_i , $i = 1, 2$, and suppose $L_1(x_1) = L_2(x_2)$. If L_1/K is the function field of a variety of general type of characteristic 0, then $L_1 = L_2$.*

PROOF. Let Y be a smooth projective variety over K whose function field is L_1 . Since the characteristic of K is 0 such a model exists. There is 1-1 correspondence between K -endomorphisms of L_1 and separable dominant rational maps from Y onto Y . [3, Theorem 2, p. 286] shows that the set of such maps is finite and hence L_1/K is rigid. Corollary 3 now applies.

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