SUBNORMAL SUBGROUPS IN $U(\mathbf{Z}G)$

JAIRO GONÇALVES, JÜRGEN RITTER AND SUDARSHAN SEHGAL

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ABSTRACT. Let U be the unit group of the integral group ring of a finite group G . We prove that every subgroup of U containing G and almost subnormal in U contains a noncyclic free group unless G is abelian or a Hamiltonian 2-group.

1. Introduction. Let UZG be the unit group of the integral group ring $\mathbf{Z}G$ of a finite group G . It is a well known result of Hartley and Pickel [4] that UZG contains a noncyclic free subgroup unless G is abelian or a Hamiltonian 2-group. A nonabelian group is said to be Hamiltonian if all its subgroups are normal. It turns out [2] that a nonabelian Hamiltonian group is isomorphic to $K_8 \times E \times O$ where E is an elementary abelian 2-group, O is an odd abelian group and K_8 is the quaternion group

$$K_8 = \{i, j | i^2 = z = j^2, z^2 = 1, i^j = i^3\}.$$

It is known that $U(K_8 \times E) = \pm K_8 \times E$ [10]. Let us say that a subgroup M of U is "almost subnormal in U " if there exists a chain of subgroups

$$M = M_0 \triangleleft M_1 \triangleleft M_2 \triangleleft \cdots \triangleleft M_s = T <_f U$$

with M_i normal in M_{i+1} and $(U : T)$ finite. We prove the

THEOREM. *Every almost subnormal subgroup of UZG containing G contains a noncyclic free group unless G is abelian or a Hamiltonian 2-group.*

We shall prove the theorem by induction on $|G|$. We shall have to handle groups all whose proper subgroups and factor groups are abelian or Hamiltonian 2-groups. These groups and their representations will be studied in §2. The proof of the theorem will be given in §3.

2. Groups with abelian or Hamiltonian factors and subgroups. In this section we characterize those finite groups G which have all proper subgroups and factor groups abelian or Hamiltonian 2-groups. We shall write $K_8 \times E$ for a Hamiltonian 2-group even though E might change from group to group. We need the following result [2, p. 187].

LEMMA 2.1. *Let G be a nonabelian p -group having a cyclic subgroup of index p . Then G is isomorphic to one of the following:*

- (i) $\langle a, b | a^{p^n} = 1 = b^p, a^b = a^{1+p^{n-1}} \rangle, n \geq 2, p$ any prime,
- (ii) *generalized quaternion group* $\langle a^{2^n} = 1, b^2 = a^{2^{n-1}}, a^b = a^{-1} \rangle, n \geq 2,$

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- (iii) the dihedral group $\langle a^{2^n} = 1 = b^2, a^b = a^{-1} \rangle, n \geq 2,$
- (iv) $\langle a, b | a^{2^n} = 1 = b^2, a^b = a^{-1+2^{n-1}} \rangle, n \geq 2.$

We remark that if $p = 2$ and $n = 2$ in (i) we get D_4 the dihedral group of order 8 of (iii) and by taking $n = 2$ in (ii) we get K_8 .

COROLLARY 2.2. *Let G be a nonabelian p -group having a cyclic subgroup of index p and having all proper factor groups abelian. Then G must be one of the following:*

- (i) $\langle a, b | a^{p^n} = 1 = b^p, a^b = a^{1+p^{n-1}} \rangle, n \geq 2, p$ any prime,
- (ii) K_8 , the quaternion group of order 8.

PROOF. In cases (ii), (iii) and (iv) of (2.1) if we factor by $\langle a^{2^{n-1}} \rangle$ we get a nonabelian group unless $n = 2$ in which case G is K_8 or D_4 . Since D_4 is included in (i) the corollary is proved.

THEOREM 2.3. *Suppose that G is a nonabelian group all whose proper subgroups and factor groups are abelian or Hamiltonian 2-groups. Then G is one of the following types:*

- I. $\langle a^{p^n} = 1 = b^p, a^b = a^{1+p^{n-1}} \rangle, n \geq 2, p$ any prime;
- II. $\langle a^p = 1 = b^p = z^p | [a, b] = z, za = az, zb = bz \rangle$ of order p^3, p odd;
- III. $A \rtimes X$, with A elementary abelian p -group (p any prime), and X of prime order $q \neq p$, acting faithfully irreducibly on A ;
- IV. $K_8 \times \langle x \rangle, O(x)$ an odd prime p ;
- V. $K_8 \times E$, a Hamiltonian 2-group.

PROOF. (a) In view of Theorem 6.5.7 of [8, p. 148], G is either III or IV or it is a p -group. Suppose that G is a p -group with every proper subgroup and factor group abelian. Then C , the centre of G , is cyclic as otherwise $C \supseteq \langle c_1 \rangle \times \langle c_2 \rangle$ with $O(c_1) = O(c_2) = p$ and the derived group $G' \subseteq \langle c_1 \rangle \cap \langle c_2 \rangle = 1$ which makes G abelian. Thus we have $G' = \langle z \rangle, z \in C, O(z) = p, C$ cyclic. Therefore, we can write $G = \langle x, y \rangle, x^y = xz, z \neq 1$ which implies that $x^p \in C, y^p \in C, G/C$ is elementary abelian of order p^2 . We have two cases:

- (i) Neither $\langle x \rangle$ nor $\langle y \rangle$ contain $\langle z \rangle$. Then $x^p = 1 = y^p$ and

$$G = \langle x^p = 1 = y^p, x^y = xz, z^p = 1 \rangle$$

is of order p^3 which is II of the theorem.

- (ii) Suppose $\langle x \rangle \supset \langle z \rangle$. Then $x^{p^n} = z^i, x^{p^{n+1}} = 1, x^y = xz$ for some $i \geq 1$.

If $y^p \notin \langle z \rangle$ let $y^{p^m} \in \langle z \rangle, y^{p^{m-1}} \notin \langle z \rangle, m > 1$. Since $y^{p^{m-1}}$ is central, factoring by $\langle y^{p^{m-1}} \rangle$ we get a nonabelian group. Thus $y^p \in \langle z \rangle$ and G has a cyclic subgroup $\langle x \rangle$ of index p . It follows by (2.2) that G is of type I or is K_8 .

We have proved that if G is a p -group for an odd prime p then G is I or II.

- (b) Now let G be a 2-group. We consider the following cases separately.

- (i) Every proper subgroup and factor group of G is abelian.
- (ii) G has a nonabelian subgroup of index 2.
- (iii) G has a nonabelian factor group.

Case (i) has already been considered in (a) where G turns out to be of type I or V.

Case (ii). We have $G \triangleright E \times K_8 \supset \langle z \rangle \supset 1$ where $\langle z \rangle$ is the centre of K_8 and $(G : E \times K_8) = 2$. Thus $\langle z \rangle \triangleleft G$ and every subgroup of $G/\langle z \rangle$ is normal as $G/\langle z \rangle$ is abelian or Hamiltonian. Therefore $K_8 \triangleleft G$. Moreover, for $x \in G$, $x \notin E \times K_8$, $(x, E) = 1$. This may be seen as follows:

$$\langle E, x, z \rangle = \{ex^\mu z^\nu | e \in E, 0 \leq \mu < 4, \nu = 0, 1\},$$

due to the fact that e being of order 2 is central in $G/\langle z \rangle$ and $O(x^4)$ divides 2. We have at most $|E| \cdot 4 \cdot 2$ elements in $\langle E, x, z \rangle$ whereas $|G| = |E| \cdot 8 \cdot 2$. Thus $\langle E, x, z \rangle \neq G$ and $\langle e \rangle \triangleleft \langle E, x, z \rangle$ which implies $ex = xe$.

The following possibilities arise:

- (1) $O(x) = 4$, $x^2 \notin \langle z \rangle$, (2) $O(x) = 2$, (3) $x^2 = z$, (4) $x^4 = z$.

Since in $G/\langle z \rangle$ every subgroup is normal, we have

$$i^x = i \text{ or } iz, \quad j^x = j \text{ or } jz, \quad (ij)^x = ij \text{ or } ijz.$$

We can rename the generators of K_8 so that $i^x = iz$, $j^x = j$.

(1) $(jx)^2 = j^2 x^2 = zx^2$ and so $O(jx) = 4$. We have $i^{jx} = (iz)^x = i$, $j^{jx} = j$. Thus we have $G \supseteq K_8 \times \langle y \rangle$ with $y = jx$ having order 4. This subgroup does not satisfy the hypothesis as $\langle iy \rangle$ is not normal.

(2) $O(x) = 2$, $i^x = iz = i^3$. Then G contains $\langle i, x \rangle$ a dihedral group, which cannot be.

(3) $x^2 = z$, $j^x = j$, $(jx)^2 = z^2 = 1$ and we have $G \supset \langle i, jx \rangle$ a dihedral group of order 8, which cannot be.

(4) $x^4 = z$, $O(x) = 8$, then $\langle K_8, x \rangle$ has a cyclic subgroup of order 8. Then by (2.1) we have (i) through (iv). But (ii), (iii) and (iv) have D_4 as factor group which is not allowed. Thus we are left with the possibility I.

Case (iii). G has a nonabelian factor group G/N . Pick $z_1 \in N$, central in G of order 2. Then $G/\langle z_1 \rangle$ is nonabelian and thus isomorphic to $E \times K_8$. We have $G/\langle z_1 \rangle = \langle \bar{i}, \bar{j} \rangle$. Either i^4 or $j^4 = z_1$, then we have a cyclic subgroup of order 8 and we are done as in (4) above or $i^4 = 1 = j^4$ and we have $G = K_8 \times E$ which is V .

THEOREM 2.4. *For the groups G of type I, II, and III of Theorem 2.3, there exists a projection ρ of $\mathbf{Q}G$ onto a simple component $K_{n \times n}$, $n \geq 2$ of $\mathbf{Q}G$ such that $\rho(G) \subset GL_n(O)$, the general linear group over O , where O is the ring of algebraic integers of an algebraic number field K . Moreover,*

(i) *if $n = 2$, $K \neq \mathbf{Q}$ or $\mathbf{Q}(\sqrt{-d})$ for some $d > 0$, then there exists a $g \in G$ such that $\rho(g)$ is noncentral and has determinant one.*

(ii) *if $n = 2$, $K = \mathbf{Q}$ or $\mathbf{Q}(\sqrt{-d})$ for some $d > 0$, then there exists a $g \in G$ such that $\rho(g) \in GL_2(\mathbf{Z})$ and is noncentral.*

PROOF. (a) Let G be of type I, $p \geq 2$, $G = \langle x^{p^n} = 1 = y^p, x^y = x^{1+p^{n-1}} \rangle$. Then consider $M = \mathbf{Q}(\zeta)$, ζ a primitive p^n th root of unity, as a G -module by letting x act as multiplication by ζ and y as the automorphism $\zeta \xrightarrow{\lambda} \zeta^{1+p^{n-1}}$. Then M is an irreducible $\mathbf{Q}G$ -module. Let K be the fixed field of λ . Then since $\text{Hom}_{\mathbf{Q}G}(M, M) \simeq K$ it is easy to see that the action of G on M yields an epimorphism $\rho: \mathbf{Q}G \rightarrow (K)_{p \times p}$ with $\mathbf{Z}G \rightarrow (O)_{p \times p}$ where O is the ring of algebraic integers of K . Clearly, $\rho(G)$ is noncentral.

(ii) Now suppose that $p = 2$ and K is not rational or an imaginary quadratic field. Since G is nonabelian $G/\zeta(G)$ is not cyclic. Thus the kernel of $\det \rho$ is not central and we can find a noncentral $g \in G$ with $\det \rho(g) = 1$.

(iii) The remaining possibility to consider is $p = 2$, $K = \mathbf{Q}$ or imaginary quadratic. If $K = \mathbf{Q}$ we are done by the first part. Now let K be imaginary quadratic. Then $[\mathbf{Q}(\zeta) : \mathbf{Q}] = 4$ and thus $n = 3$, $x^8 = 1 = y^2$, $x^y = x^5$. In this case ζ satisfies $\zeta^4 + 1 = 0$ and $\zeta^5 = -\zeta$. Then

$$\rho(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the basis $1, \zeta$ and we are done.

(b) G of Type II of order p^3 , p odd given by

$$\langle x^p = 1 = y^p = z^p, [x, y] = z, zx = xz, yz = zy \rangle.$$

Let $A = \langle x, z \rangle$. Define a character λ of A by $\lambda(z) = \zeta$, $\lambda(x) = 1$, where ζ is a primitive p th root of unity. Then $\chi = \text{ind}_A^G \lambda$ is irreducible since $\lambda(x^{y^i}) = \lambda(xz^i) = \zeta^i \neq \lambda(x)$ for $0 < i < p$ [3, p. 329]. Moreover, $\mathbf{Q}(\chi) = \mathbf{Q}(\zeta)$. It follows that we have an epimorphism $\mathbf{Q}G \twoheadrightarrow \mathbf{Q}(\zeta)_{p \times p}$ with $\mathbf{Z}G \rightarrow \mathbf{Z}[\zeta]_{p \times p}$.

(c) G of Type III, $G = A \rtimes X$ where A is an elementary abelian p -group and X is of prime order q acting faithfully irreducibly on A . At first let $A = \langle a \rangle$ be cyclic of order p . Let $M = \mathbf{Q}(\zeta_p)$, ζ_p a primitive p th root of unity. Let G act on M by: a as multiplication by ζ_p and y as the automorphism it induces on A (i.e., if $a^y = a^j$ then $y \cdot \zeta_p = \zeta_p^j$). Let K be the fixed field of this automorphism. Then $[\mathbf{Q}(\zeta_p) : K] = q$. M is an irreducible module and by picking a basis of $\mathbf{Q}(\zeta)/K$ we get $\mathbf{Q}G \xrightarrow{p} (K)_{q \times q}$ with $\mathbf{Z}G \rightarrow (O)_{q \times q}$. Suppose $q = 2$, $K \neq \mathbf{Q}$ or imaginary quadratic; we have a noncentral g such that $\det \rho(g) = 1$. Suppose that $q = 2$ and K is imaginary quadratic. Then $[\mathbf{Q}(\zeta_p) : \mathbf{Q}] = 4$ and $p = 5$ but this is not possible as $\mathbf{Q}(\zeta_p)$ has only the real subfield of degree 2.

Now, let us assume that A is not cyclic. We claim that for every nonabelian absolutely irreducible character χ its characterfield, $\mathbf{Q}(\chi)$, is equal to $\mathbf{Q}(\zeta_p)$.

PROOF OF CLAIM. The conjugacy classes of G are $K_0 = \{1\}$, $K_i = \{ax^i | a \in A\}$, $1 \leq i \leq q-1$ and $K_a = \{a, a^x, \dots, a^{x^{q-1}}\}$ for $1 \neq a \in A$. Thus there are $q + (|A| - 1)/q$ conjugacy classes and the same number of absolutely irreducible characters of G . There are $q = |G/G'|$ abelian ones and $(|A| - 1)/q$ q -dimensional ones induced by abelian characters of A . The representations belonging to the nonabelian characters are realizable over $\mathbf{Q}(\zeta_p)$. If there was a character χ such that $\mathbf{Q}(\chi) < \mathbf{Q}(\zeta_p)$ then we would have more than $(|A| - 1)/q(p - 1)$ rational irreducible nonabelian characters of G . But their number is equal to $\mathbf{Q}$ -conjugacy classes of G [3, p. 215] where $r, s \in G$ are said to be $\mathbf{Q}$ -conjugate if r^n is conjugate to s for some n relatively prime to p . Hence the $\mathbf{Q}$ -conjugacy classes of G are

$$K_0^{\mathbf{Q}} = \{1\} = K_0, \quad K_1^{\mathbf{Q}} = \{ax^i | a \in A, 1 \leq i \leq q-1\} = \bigcup_{i=1}^{q-1} K_i,$$

$$K_a^{\mathbf{Q}} = K_a \cup K_{a^2} \cup \dots \cup K_{a^{p-1}} \quad \text{for } 1 \neq a \in A.$$

Thus there are $2 + (|A| - 1)/q(p - 1)$ $\mathbf{Q}$ -conjugacy classes in G . We have two abelian rational representations and hence there are $(|A| - 1)/q(p - 1)$ nonabelian ones. Thus we have proved that $\mathbf{Q}(\chi) = \mathbf{Q}(\zeta_p)$.

It follows by counting dimensions that

$$\mathbb{Q}G = \mathbb{Q} \oplus \mathbb{Q}(\zeta_q) \oplus \frac{|A| - 1}{q(p - 1)} \mathbb{Q}(\zeta_p)_{q \times q}$$

and since the representations are induced from abelian characters

$$\mathbb{Z}G \rightarrow \mathbb{Z} \oplus \mathbb{Z}[\zeta_q] \oplus \frac{|A| - 1}{q(p - 1)} \mathbb{Z}[\zeta_p]_{q \times q}.$$

3. Proof of theorem. For a commutative ring R and an integer $n \geq 2$, we denote by $GL(n, R)$ and $SL(n, R)$ the general linear and special linear groups of degree n over R . Let $E(n, R)$ be the group generated by elementary matrices $I + rE_{ij}$, $r \in R$ where E_{ij} is the matrix with r in the (i, j) place $i \neq j$ and zeros elsewhere. If J is an ideal of R we denote by $E(n, J)$ the normal subgroup of $E(n, R)$ generated by the J -elementary matrices $I + rE_{ij}$, $r \in J$ unless $n = 2$ in which case it is the normal subgroup of $SL(2, R)$ generated by the J -elementary matrices. We need two results due to Bak [1] and Serre [9].

THEOREM 3.1 [1]. *Let $n \geq 3$ and $O = O_K$ be the ring of integers of an algebraic number field K . If S is a noncentral subgroup of $GL(n, O)$ normalized by some $E(n, J)$ for a nonzero ideal J of O then there exists a nonzero ideal J_1 of O such that S contains $E(n, J_1)$.*

COROLLARY 3.2. *Let $n \geq 3$ and $O = O_K$ be the ring of integers of an algebraic number field K . If M is a noncentral, almost subnormal subgroup of $GL(n, O)$ then there exists a nonzero integer $m \in \mathbb{Z}$ such that $E(n, m\mathbb{Z})$ is contained in M .*

PROOF. Let $M = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s = T < GL(n, O)$. Then $\tilde{T}$, the core of T , is normal of finite index in $GL(n, O)$. By the theorem, $T \supset \tilde{T} \supset E(n, J_1)$ for some nonzero ideal J_1 of O . Now M_{s-1} is normalized by $E(n, J_1)$ and thus there exists a nonzero ideal J_2 of O such that M_{s-1} contains $E(n, J_2)$. Repeating this procedure we find a nonzero ideal J of O such that $E(n, J) \subseteq M$. By picking an $m \in J \cap \mathbb{Z}$ we are finished.

THEOREM 3.3 [9]. *Let S be a noncentral subgroup of $SL(2, O)$ which is normalized by a subgroup N of finite index in $SL(2, O)$ where O is the ring of integers in an algebraic number field $K \neq \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$, $d > 0$. Then S contains $E(2, J)$ for some nonzero ideal J of O . Moreover, $SL(2, O)/E(2, J)$ is finite.*

COROLLARY 3.4. *Let O and K be as in the theorem above. If M is a noncentral almost subnormal subgroup of $SL(2, O)$ then M contains $E(2, J)$ for some nonzero ideal J of O .*

PROOF. Let $M = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s = T <_f SL(2, O)$ be a given chain of subgroups of $SL(2, O)$. As T is of finite index, it contains a normal subgroup $\tilde{T}$ of finite index in $SL(2, O)$. We find an ideal $J_1 \neq 0$ of O such that $E(2, J_1) < \tilde{T}$. Now, $E(2, J_1)$ normalizes M_{s-1} and is of finite index in $SL(2, O)$. Consequently, there is an ideal $J_2 \neq 0$ of O such that $E(2, J_2) \subset M_{s-1}$. Continuing this way we arrive at some $E(2, J)$, J nonzero ideal of O , contained in M .

THEOREM 3.5. *Let M be a noncentral, almost subnormal subgroup of $GL(2, \mathbf{Z})$. Then M contains a free group.*

PROOF. We are given a chain of groups

$$M = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s = T <_f G = GL(2, \mathbf{Z}).$$

Let p be an odd prime. Then the principal congruence subgroup

$$G(p) = \{A \in G \mid A \equiv I \pmod{p}\}$$

is a normal torsion free subgroup of finite index in G . Therefore, there exists an even natural number m such that

$$G(p)^m = \langle x^m \mid x \in G(p) \rangle \subset T \cap SL(2, \mathbf{Z}).$$

We assert that $G(p)^m \cap M_{s-1} \neq 1$. Otherwise, the commutator group, $[G(p)^m, M_{s-1}] \subset G(p)^m \cap M_{s-1} = 1$ and M_{s-1} commutes with the matrices $\begin{bmatrix} 1 & mp \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ mp & 1 \end{bmatrix}$. This implies that M_{s-1} is central in contradiction with the hypothesis. Thus $G(p)^m \cap M_{s-1} \neq 1$. By observing that $[M_{s-2}, G(p)^m \cap M_{s-1}] \subseteq G(p)^m \cap M_{s-2}$ we conclude that $G(p)^m \cap M_{s-2} \neq 1$. Repeating this procedure we obtain that $G(p)^m \cap M \neq 1$. Now let $\Gamma = SL(2, \mathbf{Z}) / \pm \{I\}$. Then Γ is a free product $\mathbf{Z}_2 * \mathbf{Z}_3$ of cyclic group [7, p. 139]; since the image $\overline{G(p)^m}$ of $G(p)^m$ in Γ is noncommutative torsion free it follows by the Kurosh subgroup theorem [6, p. 17] that $\overline{G(p)^m}$ is a free noncyclic. The subgroup $\overline{G(p)^m} \cap \overline{M}$ being a nontrivial subnormal subgroup is free noncyclic. This proves the theorem.

COROLLARY 3.6. *Let M be an almost subnormal subgroup of $U = U\mathbf{Z}G$. Let $\rho: \mathbf{Q}G \rightarrow \mathbf{Q}_{2 \times 2}$ be a projection onto a simple component. Suppose that $\rho(M) \cap GL(2, \mathbf{Z}) \neq \pm I$. Then M contains a noncyclic free group.*

PROOF. We are given $M = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s = T <_f U$. This implies $\rho(M) = \rho(M_0) \triangleleft \rho(M_1) \triangleleft \cdots \triangleleft \rho(T) <_f \rho(U)$. Intersecting with $GL(2, \mathbf{Z})$ we obtain $\rho(M) \cap GL(2, \mathbf{Z}) \triangleleft \rho(M_1) \cap GL(2, \mathbf{Z}) \triangleleft \cdots \triangleleft \rho(T) \cap GL(2, \mathbf{Z}) <_f \rho(U) \cap GL(2, \mathbf{Z})$.

Since $\rho(\mathbf{Z}G)$ and $\mathbf{Z}_{2 \times 2}$ are orders in $\mathbf{Q}_{2 \times 2}$ so is their intersection. Thus $\rho(U) \cap GL(2, \mathbf{Z})$ is of finite index in $GL(2, \mathbf{Z})$. Applying the theorem we deduce that $\rho(M)$ and hence M contains a free noncyclic subgroup.

3.7. PROOF OF THEOREM. We say a group G has property (*) if any almost subnormal subgroup of $U\mathbf{Z}G$ containing G contains a noncyclic free group, i.e., the existence of a sequence of groups

$$G \subset M = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s = T <_f U\mathbf{Z}G$$

implies that M contains a noncyclic free group.

We assert that if a subgroup or a factor group of G has property (*) then G has property (*). First, assume that a subgroup H of G has property (*). Then if M is almost subnormal in $U\mathbf{Z}G$ and contains G we get by intersection that $M \cap U\mathbf{Z}H$ is an almost subnormal subgroup of $U\mathbf{Z}H$ containing H . Thus $M \cap U\mathbf{Z}H$ contains a noncyclic free group and so does M . Now, suppose that G/N has property (*). We have $\mathbf{Q}G \simeq \mathbf{Q}(G/N) \oplus I$. Let Λ be a maximal order containing $\mathbf{Z}G$. Then $\Lambda = \Lambda_1 \oplus \Lambda_2$ and $(\Lambda^\times : U\mathbf{Z}G) < \infty$, where $\Lambda^\times$ is the multiplicative group of Λ . It follows that $(\Lambda^\times : T) < \infty$ and $(\Lambda_1^\times : \bar{T}) < \infty$. Therefore, $\bar{M}$, an almost subnormal

subgroup, containing G/N , of $\Lambda_1^\times$ contains a noncyclic free group. Thus M contains a noncyclic free group.

We prove the theorem by contradiction. Suppose that G is a group of smallest order which is not abelian or a Hamiltonian 2-group but does not satisfy (*). Then all proper subgroups and factor groups of G are abelian or Hamiltonian 2-groups. It follows by (2.3) that G is one of the types I through IV. Let us first assume that G is of type I, II or III. Then we know by (2.4) that we have a projection ρ of $\mathbf{Q}G \rightarrow K_{n \times n}$, $n \geq 2$, onto a simple component of $\mathbf{Q}G$ such that $\rho(G) \subset GL(n, O)$, where O is the ring of algebraic integers of an algebraic number field K . We are given a series

$$G \subset M = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s = T <_f U = U\mathbf{Z}G.$$

(a) If $n \geq 3$, we have

$$\rho(G) \subset \rho(M) \triangleleft \rho(M_1) \triangleleft \cdots \triangleleft \rho(M_s) = \rho(T) <_f \rho(U) \subset GL(n, O).$$

Since $\rho(\mathbf{Z}G)$ is an order of $K_{n \times n}$ contained in the maximal order $O_{n \times n}$ it follows that the last index is finite. Therefore, by (3.2) $\rho(M)$ and hence M contains a noncyclic free group.

(b) $n = 2$, $K \neq \mathbf{Q}$, $\mathbf{Q}(\sqrt{-d})$, $d > 0$. We have

$$\begin{aligned} \rho(G) \cap SL(2, O) &< \rho(M) \cap SL(2, O) \triangleleft \rho(M_2) \cap SL(2, O) \triangleleft \cdots \triangleleft \rho(T) \cap SL(2, O) \\ &<_f \rho(U) \cap SL(2, O) <_f SL(2, O). \end{aligned}$$

The last index is finite as in (a). Since we are given by (2.4) that there is a noncentral $\rho(g) \in SL(2, O) \cap \rho(G)$ we have a noncyclic free group in $\rho(M)$ and therefore in M by (3.4).

(c) $n = 2$, $K = \mathbf{Q}$ or $\mathbf{Q}(\sqrt{-d})$, $d > 0$. In this case we have by (2.4) a noncentral $\rho(g) \in GL(2, \mathbf{Z})$ and it follows by (3.6) that $\rho(M)$ and therefore M contains a noncyclic free group.

Now remains the case that G is of type IV which is handled by the next proposition and the proof of the theorem is complete.

PROPOSITION 3.8. *Let M be an almost subnormal subgroup of $U\mathbf{Z}G$ where $G = K_8 \times \langle x \rangle$ with $O(x)$ an odd prime p . Then M contains a noncyclic free group.*

PROOF. Write $K_8 = \langle a, b | a^4 = 1, b^2 = a^2, a^b = a^{-1} \rangle$. We follow the proof of [4, Theorem 2]. Let us denote by $\mathbf{H}$, the rational quaternions, $\mathbf{H} = \mathbf{Q} + \mathbf{Q}i + \mathbf{Q}j + \mathbf{Q}k$ with the relations $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Then we have an epimorphism $\mathbf{Q}G \rightarrow \mathbf{Q}(\zeta) \otimes_{\mathbf{Q}} \mathbf{H}$ where ζ is a primitive p th root of unity, the map being given by $x \rightarrow \zeta$, $a \rightarrow i$, $b \rightarrow j$. Then

$$\mathbf{Z}G \rightarrow \Gamma = \mathbf{Z}[\zeta] \oplus \mathbf{Z}[\zeta]i \oplus \mathbf{Z}[\zeta]j \oplus \mathbf{Z}[\zeta]k.$$

The given sequence

$$G \subset M = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_s = T <_f U = U\mathbf{Z}G$$

gives the sequence

$$\overline{G} = \overline{M} \triangleleft \overline{M}_1 \triangleleft \cdots \triangleleft \overline{M}_s = \overline{T} <_f \overline{U} \text{ in } \Gamma^\times.$$

Recall from [4] that $\alpha = (1 + \zeta^2)$ is a unit of $\mathbf{Z}[\zeta]$ and that

$$(1 + \zeta i)(1 - \zeta i) = 1 + \zeta^2 = \alpha, \quad (1 + \zeta j)(1 - \zeta j) = 1 + \zeta^2 = \alpha.$$

This implies that $(1 \pm \zeta i)$, $(1 \pm \zeta j)$ are units of $\overline{U}$. We can find a natural number m so that $u_1 = (1 - \zeta i)^m$ and $u_2 = (1 - \zeta j)^m$ are elements of $\overline{T}$. Now since $\overline{G}$ is contained in $\overline{M}$, the elements

$$j^{-1}u_1ju_1^{-1} = j^{-1}(1 - \zeta i)^mj(1 + \zeta i)^m\alpha^{-m} = (1 + \zeta i)^{2m}\alpha^{-m}$$

and $i^{-1}u_2iu_2^{-1} = (1 + \zeta j)^{2m}\alpha^{-m}$ lie in $\overline{M}_{s-1}$. Repeating this procedure we obtain that $(1 + \zeta i)^{2n}\alpha^{-n}$, $(1 + \zeta j)^{2n}\alpha^{-n}$ for some n lie in $\overline{M}$. It follows as in [4] by using [11, Proposition 3.12] that $\overline{M}$ contains a noncyclic free group.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SAO PAULO, SAO PAULO, BRAZIL

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AUGSBURG, AUGSBURG, WEST GERMANY

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, CANADA
T6G 2G1