A FINITENESS THEOREM FOR LINEAR SEMIGROUPS

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ABSTRACT. A classical theorem of Burnside on the finiteness of periodic linear groups is generalized to semigroups.

The following theorems are classical.

BURNSIDE'S THEOREM [3, (36.1)]. *A periodic group of matrices over a field of characteristic 0 of bounded period is finite.*

SCHUR'S THEOREM [3, (36.2)]. *A finitely generated periodic group of matrices over a field is finite.*

Schur's theorem has been generalized to semigroups in [5]. Burnside's theorem, however, is false for semigroups without further restrictions. In [7], Burnside's theorem was generalized to inverse semigroups. In this note we generalize the theorem further to a wider class of semigroups. The theorem is of interest since all the finiteness conditions in the literature with the sole exception of Theorem 8 of [4] assume the semigroups are finitely generated.

We assume familiarity with the elementary representation theory of semigroups (e.g., Chapter 5 of [2]). All undefined terminology and notation follows [2].

Recall that a class of representations is called *complete* if the direct sum of the representations is faithful on the semigroup algebra $K[S]$.

THEOREM. *Let S be a periodic semigroup of matrices over a field K of characteristic 0 of bounded period. If*

(1) *S has a complete set of irreducible representations*
then S is finite.

PROOF. By the remarks following the proof of the theorem of [6], S satisfies M_J , the d.c.c. on principal two-sided ideals, thus all its principal factors are completely (0)-simple or nilpotent. Let J be a regular J -class of S . Then J^0 is isomorphic to a Rees matrix semigroup $M^0(G; I, \Lambda; P)$. Further, G is finite by Burnside's theorem. Hence J^0 has a finite number of inequivalent irreducible representations. Since S satisfies M_J , it follows from the Munn-Ponizovskii theorem (5.33 in [2]) that every irreducible representation of S is an extension of an irreducible representation of a regular J -class. Again by [6], S has finitely many regular J -classes. Thus S has

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only finitely many irreducible representations. If h is an irreducible representation of S , then $h(S)$ can be shown to be finite by the argument used in the proof of Burnside's theorem in [3]. By (1) S has a complete set of irreducible representations and since by the above argument, there are only finitely many, their direct sum is faithful on the semigroup algebra $K[S]$. Thus $K[S]$ is finite dimensional and S must be finite.

The hypothesis (1) may seem excessively restrictive. However, the theorem is best possible in the following sense. For groups, by [1], (1), together with existence of a uniform upper bound on the degrees of the representations is equivalent to

(1') The semigroup algebra of S satisfies a polynomial identity over any field of characteristic 0.

However, the semigroup S of all strictly lower triangular matrices over an infinite field of characteristic 0 is periodic of bounded period, satisfies (1') and also M_J , yet is infinite.

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