THE SPECTRA OF CERTAIN ELLIPTIC OPERATORS IN R^N

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ABSTRACT. We give sufficient conditions for the spectra and essential spectra of certain classes of operators in R^N to be contained in an interval of the form $[d, \infty)$.

1. Introduction. We look at operators of the form $H = a(-\Delta)^n - b\Delta + q(x)$ in R^N , where Δ is the Laplacian, $a > 0$, $b > 0$, n is a positive integer greater than 1, $q(x)$ is a real valued function in R^N . This implies that H has a selfadjoint extension H bounded from below. For a selfadjoint operator the essential spectrum consists of those points in the spectrum which are not isolated eigenvalues of finite multiplicity [7].

The question we look at is: How do we find the spectrum $\sigma(H)$ of H and essential spectrum $\sigma_e(H)$ of H ? Since this question is extremely difficult to answer, we instead find a lower bound for $\sigma(H)$ and $\sigma_e(H)$. We shall present our main results in §2. We will prove them, preliminary lemmas and theorems in §§3 and 4.

Previous results on these questions can be found in Birman [1], Brenner [2], Glazman [3], Muller-Pfeiffer [4, 5], Naimark [6], Schecter [7, 8] and Yafaev [9].

2. Statement of results. We define $q_+(x) = \max(q(x), 0)$, $q_-(x) = \max(-q(x), 0)$, $q(x) = q_+(x) - q_-(x)$, $H_0 = a(-\Delta)^n - b\Delta + q_+(x)$, $H = a(-\Delta)^n - b\Delta + q(x)$, $\gamma \in D(H_0)$ if and only if $\gamma \in D(H)$ and $Q_{t,y} = I_1 \times I_2 \times I_3 \times \cdots \times I_N$ to be a square in R^N centered at y with edglength t . We now state our first theorem.

THEOREM 1. *Let $t > 0$ if there exists $0 < g_i \leq (a(\pi^2/4t^2)^{n-1} + b)\pi^2/4t$, $i = 1, 2, \dots, N$, such that for each $Q_{t,y} \in R^N$ $\int_{I_i} q_+(x) dx_i \geq g_i \forall i = 1, 2, \dots, N$ then H_0 has a selfadjoint extension H_0 such that $\sigma(H_0) \subset [(\max g_i)/2^N t, \infty)$*

COROLLARY 1.5. *If in addition $q_-(x) \leq (\max g_i)/2^N t - d$ ($(\max g_i)/2^N t \geq d > 0$) then H has a selfadjoint extension H such that $\sigma(H) \subset [d, \infty)$.*

THEOREM 2. *Under the hypotheses of Theorem 1, if*

$$\liminf_{|y| \rightarrow \infty} \int_{I_i} q_+(x) dx_i \geq \lambda_i, \quad i = 1, 2, \dots, N, \text{ and } C \geq 1,$$

such that $g_i \leq \lambda_i/C \leq (a(\pi^2/4t^2)^{n-1} + b)\pi^2/4t$, $i = 1, 2, \dots, N$, then H_0 has a selfadjoint extension H_0 such that $\sigma_e(H_0) \subset [(\max \lambda_i)/2^N C t - d, \infty)$.

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COROLLARY 2.5. *If in addition*

$$q_-(x) \leq (\max \lambda_i)/2^N C t - d \quad ((\max \lambda_i)/2^N C t \geq d > 0)$$

then H has a selfadjoint extension H such that $\sigma_e(H) \subset [d, \infty)$

For example,

$$H = a(-\Delta)^n - b\Delta + 1 + \sum_{i \in k} x_i^4 + \sum_{i \notin k} x_{i_1 \leq i \leq N}^2$$

where k is any nonempty proper subset of $\{1, 2, \dots, N\}$, $a, b > 0$, n positive integer greater than 1. Using $t = 1$ then

$$\sigma(H) \subset \left[\frac{13}{2^N(12)}, \infty \right) \quad \text{and} \quad \sigma_e(H) \subset \left[\left(a \left(\frac{\pi^2}{4} \right)^{n-1} + b \right) \frac{\pi^2}{2^{N+2}}, \infty \right).$$

3. Preliminary lemmas and theorems.

LEMMA 1. *Let I be an interval in R of length t and put $\|f\|_I = \int_I |f|^2 dx$ if $f = 0$ somewhere in I then there exists C_0 such that $C_0 \|f\|_I^2 \leq \|f'\|_I^2$ where $C_0 = \pi^2/4t^2$*

LEMMA 2. *For a, b nonnegative $a\|u\|^2 + b\|v\|^2 \geq \frac{1}{2} \min(a, b)\|u + v\|^2$.*

LEMMA 3. *Let Q be a square in R^2 and suppose there exists R such that*

$$([a(-\Delta)^n - b\Delta]u, u)_Q + (q_+u, u)_Q \geq R\|u\|_Q^2, \quad \text{if } q_-(x) \leq R - \mathcal{E} \quad (R \geq \mathcal{E} > 0).$$

Then

$$([a(-\Delta)^n - b\Delta]u, u)_Q + (qu, u)_Q \geq \mathcal{E}\|u\|_Q^2.$$

LEMMA 4. *Let P be a selfadjoint operator on R^N with $C_0^\infty \subset D(P)$. If there exists $R > 0$ such that $(P\varphi, \varphi) \geq \lambda\|Q\|^2 \quad \forall \varphi \in C_0^\infty$ which vanish in $Q_{R,0}$ then $\sigma_e(P) \subset [\lambda, \infty)$.*

Lemmas 1, 2 and 4 are proved in [2].

THEOREM 3. *Let $t > 0$, $y \in R^N$. If there exists*

$$0 < g_i \leq (a(\pi^2/4t^2)^{n-1} + b)\pi^2/4t$$

such that

$$\int_{I_i} q_+(x) dx_i \geq g_i, \quad i = 1, 2, \dots, N, \quad \forall x_j \notin I_j, \quad j \neq i,$$

then

- (1) $([a(-\Delta)^n - b\Delta + q_+]u, u)_{Q_{t,y}} \geq ((\max g_i)/2^N t)\|u\|_{Q_{t,y}}^2$
- (2) *If in addition $q_-(x) \leq (\max g_i)/2^N t - d \quad ((\max g_i)/2^N t \geq d > 0)$ then*

$$([a(-\Delta)^n - b\Delta + q]u, u)_{Q_{t,y}} \geq d\|u\|_{Q_{t,y}}^2.$$

PROOF. Let $Q = Q_{t,y}$, $y \in R^N$, $Q = I_1 \times I_2 \times \dots \times I_N$, $D_k^i u = \partial^i u / \partial x_k^i$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots, N$,

$$\bar{x}_i = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \quad d\bar{x}_i = dx_1 dx_2 \dots dx_{i-1} dx_{i+1} \dots dx_N,$$

$$L_i = I_1 \times I_2 \times \dots \times I_{i-1} \times I_{i+1} \times \dots \times I_N, \quad u_i^2(\bar{x}_i) = \min_{x_i \in I_i} u^2(x_i, \bar{x}_i).$$

Then

$$\|u_i\|_{I_i}^2 = \int_{I_i} |u_i|^2 dx_i = t u_i^2 \Rightarrow u_i^2 = \frac{\|u_i\|_{I_i}^2}{t}, \quad i = 1, 2, \dots, N.$$

Then

$$\begin{aligned} & ([a(-\Delta)^n - b\Delta + q_+(x)]u, u)_Q \\ & \geq \int_Q a|D_1^n u|^2 + a|D_2^n u|^2 + \dots + a|D_N^n u|^2 + b|D_1^1 u|^2 \\ & \quad + b|D_2^1 u|^2 + \dots + b|D_N^1 u|^2 + q_+ u dx. \end{aligned}$$

I. First D_1^n and D_1^1 then D_2^n and $D_2^1, \dots, D_N^n$ and D_N^1

$$\begin{aligned} & \geq \int_{L_1} \int_{I_1} a|D_1^n(u \pm u_1)|^2 + b|D_1^1(u \pm u_1)|^2 + q_+ u_1^2 dx_1 d\bar{x}_1 \\ & \quad + \int_Q a|D_2^n u|^2 + \dots + a|D_N^n u|^2 + b|D_2^1 u|^2 + \dots + b|D_N^1 u|^2 dx. \end{aligned}$$

Either $u + u_1$ or $u - u_2$ will vanish somewhere in I_1 .

$$\begin{aligned} & \geq \int_{L_1} \int_{I_1} aC_0^{n-1}|D_1^1(u \pm u_1)|^2 + b|D_1^1(u \pm u_1)|^2 + q_+ u_1^2 dx_1 d\bar{x}_1 + \\ & \quad \text{[Lemma 1 } (n-1) \text{ times]} \\ & = \int_{L_1} \int_{I_1} (aC_0^{n-1} + b)|D_1^1(u \pm u_1)|^2 + q_+ u_1^2 dx_1 d\bar{x}_1 + \\ & \geq \int_{L_1} (aC_0^{n-1} + b)C_0\|u \pm u_1\|_{I_1}^2 + u_1^2 \int_{I_1} q_+(x) dx_1 d\bar{x}_1 + \text{[Lemma 1]} \\ & \geq \int_{L_1} (aC_0^{n-1} + b)C_0\|u \pm u_1\|_{I_1}^2 + u_1^2 g_1 d\bar{x}_1 + \\ & = \int_{L_1} (aC_0^{n-1} + b)C_0\|u \pm u_1\|_{I_1}^2 + \frac{g_1}{t}\|u_1\|_{I_1}^2 d\bar{x}_1 + \\ & \geq \int_{L_1} \frac{g_1}{2t}\|u\|_{I_1}^2 d\bar{x}_1 \quad \text{[Lemma 2]} \\ & = \int_{L_2} \int_{I_2} \frac{g_1}{2t}|u|^2 + a|D_2^n u|^2 + b|D_2^1 u|^2 dx_2 d\bar{x}_2 \\ & \quad + \int_Q a|D_3^n u|^2 + \dots + a|D_N^n u|^2 + b|D_3^1 u|^2 + \dots + b|D_N^1 u|^2 dx \quad \text{[step 1]} \\ & \geq \int_{L_2} \int_{I_2} \frac{g_1}{2t}u_2^2 + a|D_2^n(u \pm u_2)|^2 + b|D_2^1(u \pm u_2)|^2 dx_2 d\bar{x}_2 + \text{[step 2]} \end{aligned}$$

Either $u + u_2$ or $u - u_2$ will vanish somewhere in I_2 .

$$\begin{aligned}
 &\geq \int_{L_2} \int_{I_2} \frac{g_1}{2t} u_2^2 + aC_0^{n-1} |D_2^1(u \pm u_2)|^2 + b|D_2^1(u \pm u_2)|^2 dx_2 d\bar{x}_2 + \\
 &\quad \text{[Lemma 1 } (n-1) \text{ times] [step 3]} \\
 &= \int_{L_2} \int_{I_2} (aC_0^{n-1} + b) |D_2^1(u \pm u_2)|^2 + \frac{g_1}{2t} u_2^2 dx_2 d\bar{x}_2 + \text{[step 4]} \\
 &\geq \int_{L_2} (aC_0^{n-1} + b) C_0 \|u \pm u_2\|_{I_2}^2 + \frac{g_1}{2} u_2^2 d\bar{x}_2 + \text{[Lemma 1] [step 5]} \\
 &= \int_{L_2} (aC_0^{n-1} + b) C_0 \|u \pm u_2\|_{I_2}^2 + \frac{g_1}{2t} \|u_2\|_{I_2}^2 d\bar{x}_2 + \text{[step 6]} \\
 &\geq \int_{L_2} \frac{g_1}{4t} \|u\|_{I_2}^2 d\bar{x}_2 \quad \text{[Lemma 2] [step 7]}
 \end{aligned}$$

Repeat steps 1–7 for L_j I_j , $j = 3, 4, \dots, N$. Then

$$\int_{L_N} \frac{g_1}{2^N t} \|u\|_{I_N}^2 d\bar{x}_N = \frac{g_1}{2^N t} \|u\|_Q^2.$$

II. D_1^n and D_1^1 and any order of D_2^n and $D_2^1, \dots, D_N^n$ and D_N^1 gives the same result.

III. All other possible ways to do D_1^n and D_1^1 , D_2^n and $D_2^1, \dots, D_N^n$ and D_N^1 with D_i^n and D_i^1 done first gives the result

$$\frac{g_i}{2^N t} \|u\|_Q^2.$$

Hence

$$([a(-\Delta)^n - b\Delta + q_+]u, u)_{Q_{t,y}} \geq \frac{\max g_i}{2^N t} \|u\|_{Q_{t,y}}^2.$$

Part (2) follows from Lemma 3.

THEOREM 4. Let $t > 0$ if there exists $0 < g_i \leq (a(\pi^2/4t^2)^{n-1} + b)\pi^2/4t$ such that for each $Q_{t,y} \in R^N$ $\int_{I_i} q_+(x) dx_i \geq g_i$, $i = 1, 2, \dots, N$, $\forall x_j \in I_j$, $j \neq i$. Then

- (1) $([a(-\Delta)^n - b\Delta + q_+]u, u)_{Q_{t,y}} \geq ((\max g_i)/2^N t) \|u\|_{Q_{t,y}}^2$.
- (2) If in addition $q_-(x) \leq (\max g_i)/2^N t - d$ ($(\max g_i)2^N t \geq d > 0$), then

$$([a(-\Delta)^n - b\Delta + q]u, u)_{Q_{t,y}} \geq d \|u\|_{Q_{t,y}}^2.$$

PROOF. Follows from Theorem 3.

4. Proofs of main results.

Proof of Theorem 1. By Theorem 4 for each $Q_{t,y}$

$$([a(-\Delta)^n - b\Delta + q_+]u, u)_{Q_{t,y}} \geq \frac{\max g_i}{2^N t} \|u\|_{Q_{t,y}}^2 \quad \forall y \in R^N.$$

If we cover E^N with such squares and sum, $((\max g_i)/2^N t) \|u\|^2 \leq (H_0 u, u) \Rightarrow \sigma(H_0) \subset [(\max g_i)/2^N t, \infty)$ [8].

Proof of Corollary 1.5. By Theorem 4(2) for each $Q_{t,y}$

$$([a(-\Delta)^n - b\Delta + q]u, u)_{Q_{t,y}} \geq d \|u\|_{Q_{t,y}}^2 \quad \forall y \in R^N.$$

If we cover E^N with such squares and sum

$$d\|u\|^2 \leq (Hu, u) \Rightarrow \sigma(H) \subset [d, \infty) \quad [8].$$

Proof of Theorem 2. Let $\varepsilon > 0$ be given and take R so large that $\int_{I_i} q_+(x) \geq (\lambda_i - \varepsilon)/C\forall i$, $i = 1, 2, \dots, N$, for y outside the square $Q_{R,0}$.

Then by Theorem 3

$$\begin{aligned} ([a(-\Delta)^n - b\Delta + q_+]u, u)_{Q_{t,y}} &\geq \frac{\max(\lambda_i - \varepsilon)}{2^N C t} \|u\|_{Q_{t,y}}^2 \quad \text{for such } y \\ \Rightarrow \sigma_e(H_0) &\subset \left[\frac{\max(\lambda_i - \varepsilon)}{2^N C t}, \infty \right) \quad [\text{Lemma 4}]. \end{aligned}$$

Since this is true for every $\varepsilon \geq 0$

$$\Rightarrow \sigma_e(H_0) \subset \left[\frac{\max \lambda_i}{2^N C t}, \infty \right)$$

Proof of Corollary 2.5. Let $\varepsilon > 0$ be given, $q_-(x) \leq (\max \lambda_i)/2^N c t - (d - \varepsilon)$. Then by Theorem 3(2)

$$\begin{aligned} ([a(-\Delta)^n - b\Delta + q]u, u)_{Q_{t,y}} &\geq (d - \varepsilon) \|u\|_{Q_{t,y}}^2 \quad \text{for } y \text{ outside } Q_{R,0} \\ \Rightarrow \sigma_e(H) &\subset [d - \varepsilon, \infty) \quad [\text{Lemma 4}]. \end{aligned}$$

Since this is true for every $\varepsilon \geq 0$

$$\Rightarrow \sigma_e(H) \subset [d, \infty).$$

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