THE AREA OF THE COMPLEMENT OF A CONFORMALLY RIGID DOMAIN

MARIUS OVERHOLT

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ABSTRACT. A hyperbolic subdomain D of $\hat{\mathbb{C}}$ is said to be (conformally) rigid if any conformal map from D into $\hat{\mathbb{C}}$ is either a Möbius transformation, or has Schwarzian norm larger than a positive constant depending on D only. We show that the complement of a conformally rigid domain has zero area.

1. Introduction. Let D be a hyperbolic subdomain (not necessarily simply connected) of $\hat{\mathbb{C}}$; i.e., D has the unit disk $\mathbf{D}$ as its universal covering surface. Then D carries a metric of constant negative curvature, called the Poincaré metric on D . We shall denote its density by $\rho_D(z)$. It is inherited from the Poincaré density

$$\rho_{\mathbf{D}}(z) = (1 - |z|^2)^{-1}$$

of the unit disk, via the covering projection. The hyperbolic sup-norm of weight -2 on D is given by

$$\|\phi\|_D = \sup_{z \in D} |\phi(z)| \rho_D(z)^{-2}.$$

Further, if $f: D \rightarrow \hat{\mathbb{C}}$ is a locally univalent meromorphic function, then the Schwarzian derivative

$$S(f) = (f''/f')' - \frac{1}{2}(f''/f')^2$$

is a holomorphic function on D , and we may form the Schwarzian norm $\|S(f)\|_D$ of f on D . It may be infinite, of course.

The following definition is due to Thurston [5].

DEFINITION (THURSTON). A hyperbolic domain $D \subseteq \hat{\mathbb{C}}$ is (conformally) rigid if there exists a constant $s > 0$ such that if $f: D \rightarrow \hat{\mathbb{C}}$ is conformal and $\|S(f)\|_D \leq s$, then f is a Möbius transformation.

In this definition, and elsewhere in this paper, conformal is to mean holomorphic and globally injective.

In [5] Thurston proved the existence of rigid domains by construction. His examples are complements of quasicircles, and so the complements of these rigid domains automatically have zero area. Astala [1] gives an alternative proof for the existence of rigid domains. His examples are contained among those of Thurston.

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2. Complements of rigid domains. In the following, $m(A)$ will always denote the Lebesgue planar measure of a set A . We shall need the following theorem due to Nguyen Xuan Uy [4].

THEOREM (NGUYEN). *If Γ is compact in $\mathbb{C}$, with $m(\Gamma) > 0$, then there exists a nonconstant bounded holomorphic Lipschitz function on $\hat{\mathbb{C}} \setminus \Gamma$.*

In addition, we shall need the following result due to Beardon and Gehring [2].

THEOREM (BEARDON AND GEHRING). *If D is a hyperbolic subdomain of $\hat{\mathbb{C}}$, then*

$$|S(f, z)| \leq 12\rho_D(z)^2$$

in D , for any conformal map f from D into $\hat{\mathbb{C}}$.

We may now prove the following

THEOREM. *Let D be a conformally rigid domain, and $\Gamma = \hat{\mathbb{C}} \setminus D$. Then $m(\Gamma) = 0$.*

PROOF. We may without loss of generality assume that $\infty \in D$. We shall assume that $m(\Gamma) > 0$, and conclude that D is not conformally rigid.

Since Γ is compact with $m(\Gamma) > 0$, Nguyen's theorem gives a nonconstant bounded holomorphic Lipschitz function F on D . For convenience, we write

$$G = F'F''' - \frac{3}{2}F''^2.$$

Further, let $A = \text{Lip}(F)$. Now put

$$f(z) = z + \varepsilon F(z),$$

with $|\varepsilon| < 1/A$. Then f is conformal on D . Note that f is not a Möbius transformation unless $\varepsilon = 0$. We have

$$S(f) = (\varepsilon F''' + \varepsilon^2 G)/(1 + \varepsilon F')^2.$$

Now since F satisfies a Lipschitz condition, the inequality $|F'(z)| \leq A$ is valid in D . Furthermore, we have

$$|S(f, z)| \leq 12\rho_D(z)^2$$

in D by the theorem of Beardon and Gehring. Thus

$$|\varepsilon F'''(z) + \varepsilon^2 G(z)| \leq |1 + \varepsilon F'(z)|^2 12\rho_D(z)^2 \leq 12(1 + A|\varepsilon|)^2 \rho_D(z)^2$$

in D . Putting $\varepsilon = \pm 1/(2A)$ gives

$$|\pm F'''(z)/2A + G(z)/4A^2| \leq 27\rho_D(z)^2$$

and this yields

$$|F'''(z)| \leq 54A\rho_D(z)^2 \quad \text{and} \quad |G(z)| \leq 108A^2\rho_D(z)^2$$

by the triangle inequality. But then

$$\begin{aligned} |S(f, z)|\rho_D(z)^{-2} &\leq \frac{|\varepsilon||F'''(z)| + |\varepsilon|^2|G(z)|}{(1 - |\varepsilon||F'(z)|)^2}\rho_D(z)^{-2} \\ &\leq \frac{|\varepsilon|54A\rho_D(z)^2 + |\varepsilon|^2108A^2\rho_D(z)^2}{(1 - |\varepsilon|A)^2}\rho_D(z)^{-2} \\ &= \frac{54A|\varepsilon| + 108A^2|\varepsilon|^2}{(1 - A|\varepsilon|)^2} \end{aligned}$$

and so

$$\|S(f)\|_D = \sup_{z \in D} |S(f, z)| \rho_D(z)^{-2} \leq \frac{54A|\varepsilon| + 108A^2|\varepsilon|^2}{(1 - A|\varepsilon|)^2}.$$

In particular, $\|S(f)\|_D$ tends to zero as ε tends to zero, and thus D is not a rigid domain.

3. Remarks. From the above, we see that if $S(f)$ is an isolated point of the set of Schwarzians of conformal maps $f: \mathbf{D} \rightarrow \hat{\mathbf{C}}$, in the topology induced by the norm $\|\cdot\|_{\mathbf{D}}$, then the omitted set of f has zero area. For the details of the connection between rigid domains and isolated Schwarzians, see [1 or 5].

The device of using Nguyen's theorem to produce a conformal map was used by Hamilton in [3] to identify the extreme points of an important class of univalent functions.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109

Current address: Department of Mathematics, Indiana University, Bloomington, Indiana 47405