A COMPARISON THEOREM FOR LINEAR DIFFERENCE EQUATIONS

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ABSTRACT. Let a be real, $I = \{a, a + 1, \dots, b\}$ where $b - a$ is a positive integer or $I = \{a, a + 1, \dots\}$. Let n be a positive integer and let $I^n = \{a, a + 1, \dots, b + n\}$ if $b < \infty$ or $I^n = I$ otherwise. Consider the n th order difference equation $Pu(m) = \sum_{j=0}^n \alpha_j(m)u(m+j) = 0$, $\alpha_n(m) = 1$, $\alpha_0(m) \neq 0$, $m \in I$. It is shown that if $0 \leq r(m) \leq q(m)$, $m \in I$ and if the equations $Pu(m) = 0$ and $Pu(m) + q(m)u(m) = 0$ are disconjugate on I^n , then the equation $Pu(m) + r(m)u(m) = 0$ is disconjugate on I^n .

Let a be real and let $I = \{a, a + 1, \dots, b\}$ where $b - a$ is a positive integer or $I = \{a, a + 1, \dots\}$. Let n be a positive integer and define $I^n = \{a, a + 1, \dots, b + n\}$ if $b < \infty$ or $I^n = I$ otherwise. Consider the linear n th order difference equation

$$(1) \quad Pu(m) = \sum_{j=0}^n \alpha_j(m)u(m+j) = 0,$$

where $\alpha_n(m) = 1$, $\alpha_0(m) \neq 0$, $m \in I$ and the coefficients $\alpha_0, \dots, \alpha_n$ are defined on I . In a remark preceding Theorem 1, we shall indicate why the assumption that $\alpha_0(m)$ does not vanish for any $m \in I$ is made. The object of this paper is to show that if $Pu(m) = 0$ and $Pu(m) + q(m)u(m) = 0$ are disconjugate on I^n , where $0 \leq q(m)$ for $m \in I$, then if $0 \leq r(m) \leq q(m)$, $m \in I$, the equation $Pu(m) + r(m)u(m) = 0$ is disconjugate on I^n . This is stated and proved in Theorem 4.

Oscillatory behavior of solutions and comparison theorems for difference equations have been of interest in recent years. For example, see [1, 3, 4, 6-8]. Many of these authors have considered equations of the form

$$Pu(m) + q_{n-1}(m)u(m+n-1) + \dots + q_1(m)u(m+1) = 0.$$

Note that in these references the coefficient $q_0(m)$ for the lowest order term $u(m)$ is identically zero on I . In this paper, we adapt techniques employed by Nehari [5] for differential operators in order to handle the case $q_0(m) \geq 0$ on I .

We begin with the definition of the disconjugacy of (1) on I^n and state criteria equivalent to the disconjugacy of (1) on I^n due to Hartman [2]. For a finite or infinite sequence of real numbers $u = u(a), u(a+1), \dots$, we say that $m = a$ is a *generalized zero* for u if $u(a) = 0$ and $m > a$ is a *generalized zero* for u if $u(m) = 0$ or

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there is an integer $k \geq 1$ such that $m - k \geq a$, $(-1)^k u(m - k)u(m) > 0$ and if $k > 1$, $u(m - k + 1) = \cdots = u(m - 1) = 0$. The difference equation (1) is *disconjugate on* I^n if no solution $u \not\equiv 0$ has n generalized zeros on I^n .

Let $u_1, \dots, u_n$ be n real-valued functions defined on I^n . Let $k \in \{1, \dots, n\}$, let $1 \leq r_1 < \cdots < r_k \leq n$, and define

$$(2) \quad W(r_1, \dots, r_k)(m) = \det\{u_{r_i}(m + j)\}, \quad 1 \leq i, j \leq k.$$

Let $\Delta^0 u(m) \equiv u(m)$, $\Delta u(m) = u(m + 1) - u(m)$ and $\Delta^j u(m) = \Delta(\Delta^{j-1} u(m))$, $j \geq 2$. Note that by determinant properties and elementary row operations

$$(3) \quad W(r_1, \dots, r_k)(m) = \det\{\Delta^{j-1} u_{r_i}(m)\}, \quad 1 \leq i, j \leq k.$$

REMARK. Determinant identities are employed throughout this paper; in particular, nonzero determinants play a vital role in this development. It is here that we require $\alpha_0(m) \neq 0$ for any $m \in I$. For example, by (1) and (2) it readily follows that if $u_1, \dots, u_n$ are n solutions of (1) on I^n then $W(1, \dots, n)(m + 1) = (-1)^n \alpha_0(m) W(1, \dots, n)(m)$. Thus, a fundamental result concerning the solution set of (1) is that $W(1, \dots, n)(a) \neq 0$ if and only if $W(1, \dots, n)(m) \neq 0$ for any $m \in I$. See Fort [1] for further fundamental results concerning (1).

The following theorem is due to Hartman [2, Theorem 5.1].

THEOREM 1. *The following are equivalent:*

- (i) (1) is *disconjugate on* I^n ;
- (ii) there exists a fundamental set of solutions $u_1, \dots, u_n$ of (1) on I^n such that $W(1, \dots, k)(m) > 0$ on I^{n-k+1} , $1 \leq k \leq n$.
- (iii) there exist positive functions $s_k(m)$, $m \in I^{n-k+1}$, $1 \leq k \leq n$, such that

$$(4) \quad Pu(m) = s_{n+1}(m) \Delta [s_n(m) \Delta [\cdots \Delta (s_1(m) u(m)) \cdots]], \quad m \in I,$$

where $s_{n+1} = 1/s_1 \cdots s_n$;

- (iv) $u(m) \equiv 0$ is the only solution of (1) having k (≥ 0) successive zeros at $m = a, \dots, a + k - 1$ and $n - k$ successive generalized zeros at $m = j, \dots, j + n - k - 1$, for some j , $a + k \leq j \in I^{k+1}$. (By $k = 0$, we mean that $u(a) \neq 0$ if $a < j$.)

REMARK. In the proof of $f \Rightarrow a$ in Theorem 5.1, Hartman [2] proves a stronger statement than (iv). The stronger statement is that $u(m) \equiv 0$ is the only solution of (1) having k (≥ 0) successive zeros at $m = a, \dots, a + k - 1$, and $n - k - 1$ successive zeros at $m = j, \dots, j + n - k - 2$ and a generalized zero at $m = j + n - k - 1$, for some j , $a + k \leq j \in I^{k+1}$. We shall use this stronger statement in the proof of Theorem 4.

We now introduce notation which is motivated by the factorization of P in (4) and is analogous to the notation employed by Nehari [5]. Let $Pu(m) = 0$ be disconjugate on I^n and let $s_1, \dots, s_{n+1}$ be as in (4). Define quasi-difference operators P_k by

$$P_0 u(m) = s_1(m) u(m), \quad m \in I^n,$$

$$P_k u(m) = s_{k+1}(m) (\Delta P_{k-1} u(m)), \quad m \in I^{n-k}, \quad 1 \leq k \leq n.$$

Thus, $P_n u(m) \equiv Pu(m)$. Let $u_1, \dots, u_n$ be n real-valued functions defined on I^n . Let $k \in \{1, \dots, n\}$, let $1 \leq r_1 < \cdots < r_k \leq n$, and let $0 \leq i_1 < \cdots < i_k \leq n$. For $m \in I^{n-i_k}$, define $D(i_1, \dots, i_k; r_1, \dots, r_k)(m) = \det\{P_{i_l} u_{r_l}(m)\}$, $1 \leq l, j \leq k$. Note

that by determinant properties and elementary row operations as in (3), it follows that

$$(5) \quad D(0, \dots, k-1; r_1, \dots, r_k)(m) = \prod_{j=1}^k \prod_{l=0}^{k-j} s_j(m+l) W(r_1, \dots, r_k)(m),$$

$1 \leq k \leq n$, $m \in I^{n-k+1}$. Thus, it follows that if Theorem 1(iii) holds, then

$$D(0, \dots, k-1; r_1, \dots, r_k)(m) > 0$$

if and only if $W(r_1, \dots, r_k)(m) > 0$.

The following lemma provides a discrete analogue to differentiating determinant functions. Let

$$A_k = \{\sigma : \{1, \dots, k\} \rightarrow \{0, 1\}\}.$$

LEMMA 2. Let $k \in \{1, \dots, n\}$, let $0 \leq i_1 < \dots < i_k \leq n-1$, $1 \leq r_1 < \dots < r_k \leq n$, and let $m \in I^{n-i_k-1}$. Then

$$(6) \quad \begin{aligned} & D(i_1, \dots, i_k; r_1, \dots, r_k)(m+1) \\ &= \sum_{\sigma \in A_k} \left(\prod_{j: \sigma(j)=1} \frac{1}{s_{i_j+2}(m)} \right) D(i_1 + \sigma(1), \dots, i_k + \sigma(k); r_1, \dots, r_k)(m). \end{aligned}$$

Note that if $i_j + \sigma(j) = i_{j+1} + \sigma(j+1)$, then the corresponding determinant is zero.

PROOF. The proof is by induction on k . For $k = 1$,

$$\begin{aligned} D(i_1; r_1)(m+1) &= P_{i_1} u_{r_1}(m+1) = P_{i_1} u_{r_1}(m+1) - P_{i_1} u_{r_1}(m) + P_{i_1} u_{r_1}(m) \\ &= (1/s_{i_1+2}(m)) D(i_1+1; r_1)(m) + D(i_1; r_1)(m). \end{aligned}$$

Let $k > 1$ and assume that (6) holds for each $1 \leq l \leq k-1$, $0 \leq i_1 < \dots < i_l \leq n-1$, and $1 \leq r_1 < \dots < r_l \leq n$. Employ a cofactor expansion along the k th row to obtain

$$(7) \quad \begin{aligned} & D(i_1, \dots, i_k; r_1, \dots, r_k)(m+1) \\ &= P_{i_k} u_{r_1}(m+1) D(i_1, \dots, i_{k-1}; r_2, \dots, r_k)(m+1) \\ &\quad + \dots + P_{i_k} u_{r_k}(m+1) D(i_1, \dots, i_{k-1}; r_1, \dots, r_{k-1})(m+1) \\ &= (1/s_{i_k+2} P_{i_k+1} u_{r_1}(m) + P_{i_k} u_{r_1}(m)) D(i_1, \dots, i_{k-1}; r_2, \dots, r_k)(m+1) \\ &\quad + \dots + (1/s_{i_k+2} P_{i_k+1} u_{r_k}(m) + P_{i_k} u_{r_k}(m)) \\ &\quad \times D(i_1, \dots, i_{k-1}; r_1, \dots, r_{k-1})(m+1). \end{aligned}$$

By the induction assumption

$$\begin{aligned}
 & P_{i_k} u_{r_1}(m) D(i_1, \dots, i_{k-1}; r_2, \dots, r_k)(m+1) \\
 & \quad + \dots + P_{i_k} u_{r_k}(m) D(i_1, \dots, i_{k-1}; r_1, \dots, r_{k-1})(m+1) \\
 & = P_{i_k} u_{r_1}(m) \left(\sum_{\sigma \in A_{k-1}} \left(\prod_{j: \sigma(j)=1} \frac{1}{s_{i_j+2}(m)} \right) \right. \\
 & \quad \left. \times D(i_1 + \sigma(1), \dots, i_{k-1} + \sigma(k-1); r_2, \dots, r_k)(m) \right) \\
 & \quad + \dots + P_{i_k} u_{r_k}(m) \left(\sum_{\sigma \in A_{k-1}} \left(\prod_{j: \sigma(j)=1} \frac{1}{s_{i_j+2}(m)} \right) \right. \\
 & \quad \left. \times D(i_1 + \sigma(1), \dots, i_{k-1} + \sigma(k-1); r_1, \dots, r_{k-1})(m) \right) \\
 & = \sum_{\sigma \in A_{k-1}} \left(\prod_{j: \sigma(j)=1} \frac{1}{s_{i_j+2}(m)} \right) \\
 & \quad \times D(i_1 + \sigma(1), \dots, i_{k-1} + \sigma(k-1), i_k; r_1, \dots, r_k)(m)
 \end{aligned}$$

using a cofactor expansion along the k th row. Similarly,

$$\begin{aligned}
 & P_{i_{k+1}} u_{r_1}(m) D(i_1, \dots, i_{k-1}; r_2, \dots, r_k)(m+1) \\
 & \quad + \dots + P_{i_{k+1}} u_{r_k}(m) D(i_1, \dots, i_{k-1}; r_1, \dots, r_{k-1})(m+1) \\
 & = \sum_{\sigma \in A_{k-1}} \left(\prod_{j: \sigma(j)=1} \frac{1}{s_{i_j+2}(m)} \right) \\
 & \quad \times D(i_1 + \sigma(1), \dots, i_{k-1} + \sigma(k-1), i_k + 1; r_1, \dots, r_k)(m).
 \end{aligned}$$

Thus, (6) follows from (7).

We now consider linear difference equations of the form

$$(8) \quad Pu(m) = \lambda \alpha(m) u(m),$$

where $\alpha(m) \geq 0$, $m \in I$ and $\lambda = 1$ or $\lambda = -1$.

LEMMA 3. Assume $Pu(m) = 0$ is disconjugate on I^n . Let $u_1, \dots, u_n$ be the fundamental set of solutions of (8) on I^n satisfying the initial conditions

$$(9) \quad P_{j-1} u_l(a) = \delta_{jl}, \quad 1 \leq j \leq n, \quad 1 \leq l \leq n.$$

If $\lambda = -1$, then all even order minors of $D(0, \dots, n-1; 1, \dots, n)(m)$ are nonnegative and nondecreasing on their maximal intervals of existence. If $\lambda = 1$, then all odd order minors are nonnegative and nondecreasing on their maximal intervals of existence.

PROOF. The proof is based on an argument of Nehari [5, Theorem 3.1], who credits Mikusinski [4] with the motivation of the argument. Let $0 \leq i_1 < \dots < i_k \leq n-1$ and $1 \leq r_1 < \dots < r_k \leq n$ be given. If $i_k < n-1$,

$$D(i_1, \dots, i_k; r_1, \dots, r_k)(n+1)(m+1)$$

is given by (6). If $i_k = n - 1$,

$$(10) \quad \begin{aligned} & D(i_1, \dots, i_{k-1}, n-1; r_1, \dots, r_k)(m+1) \\ &= \sum_{\sigma \in A_k} \left(\prod_{j: \sigma(j)=1} \frac{1}{s_{i_j+2}(m)} \right) D(i_1 + \sigma(1), \dots, n-1 + \sigma(k); r_1, \dots, r_k)(m). \end{aligned}$$

If $\sigma(k) = 1$, employ (8) to obtain

$$\begin{aligned} & D(i_1 + \sigma(1), \dots, i_{k-1} + \sigma(k-1), n; r_1, \dots, r_k)(m) \\ &= (-1)^{k-1} \lambda(\alpha(m)/s_1(m)) D(0, i_1 + \sigma(1), \dots, i_{k-1} + \sigma(k-1); r_1, \dots, r_k)(m). \end{aligned}$$

Thus, the relations (6) and (10) may be regarded as the first order system of difference equations $U(m+1) = A(m)U(m)$, where the components of $U(m)$ have the form $D(i_1, \dots, i_k; r_1, \dots, r_k)(m)$ and $i_1, \dots, i_k$ range over all integers satisfying $0 \leq i_1 < \dots < i_k \leq n-1$ and $r_1, \dots, r_k$ are fixed and satisfy $1 \leq r_1 < \dots < r_k \leq n$. By (6), the diagonal elements of $A(m)$ are all 1 and by (9), the components of $U(a)$ are either zeros or ones; in particular, the components of $U(a)$ are nonnegative. If $\lambda = -1$ and k is even, each element of $A(m)$ is positive. Thus, each component of $U(m)$ is nonnegative and nondecreasing.

If $\lambda = 1$ and k is odd, the result follows similarly.

We now state and prove the main result of the paper. This is a discrete version of Nehari's Theorem 4.1 [5].

THEOREM 4. *Let $0 \leq r(m) \leq q(m)$, $m \in I$, and assume the equations $Pu(m) = 0$ and $Pu(m) + q(m)u(m) = 0$ are disconjugate on I^n . Then the equation $Pu(m) + r(m)u(m) = 0$ is disconjugate on I^n .*

PROOF. First consider the equation $Pu(m) + r(m)u(m) = 0$ in the form

$$(11) \quad Pu(m) = -r(m)u(m)$$

and let $u_1, \dots, u_n$ be the fundamental set of solutions of (11) satisfying the initial conditions (9). Then by Lemma 3 all even order minors of $D(0, \dots, n-1; 1, \dots, n)(m)$ are nonnegative and nondecreasing. By (9),

$$D(0, \dots, n-k-1; k+1, \dots, n)(a+k) > 0.$$

Thus, if $n-k$ is even, $D(0, \dots, n-k-1; k+1, \dots, n)(m) > 0$, $m \geq a+k$, and so by (5),

$$(12) \quad W(k+1, \dots, n)(m) > 0, \quad m \geq a+k.$$

Now consider the equation $Pu(m) + r(m)u(m) = 0$ in the form

$$(13) \quad Mu(m) \equiv Pu(m) + q(m)u(m) = (q-r)(m)u(m).$$

$Mu(m) = 0$ is disconjugate on I^n , so there exist positive functions $\sigma_1, \dots, \sigma_{n+1}$ such that M has the factorization given by (4) where each s_j is replaced by σ_j . Define quasi-difference operators M_k as we did P_k where each s_j is replaced by σ_j . Let $\hat{u}_1, \dots, \hat{u}_n$ be the fundamental set of solutions of (13) satisfying the initial conditions

$$(14) \quad M_{j-1}\hat{u}_l(a) = \delta_{jl}, \quad 1 \leq j \leq n, \quad 1 \leq l \leq n.$$

Define

$$\hat{D}(i_1, \dots, i_k; r_1, \dots, r_k)(m) = \det\{M_{i_j} \hat{u}_{r_l}(m)\}$$

and

$$\hat{W}(r_1, \dots, r_k)(m) = \det\{\hat{u}_{r_l}(m + j)\}.$$

Then by Lemma 3 with $\lambda = 1$, all odd order minors of $\hat{D}(0, \dots, n-1; 1, \dots, n)(m)$ are nonnegative and nondecreasing; in particular, if $n-k$ is odd,

$$(15) \quad \hat{W}(k+1, \dots, n)(m) > 0, \quad m \geq a+k.$$

Assume $Pu(m) + r(m)u(m) = 0$ is not disconjugate on I^n . Then, from the remark following Theorem 1, there is some $k \geq 0$ and some solution $u \neq 0$ such that u has k successive zeros at $m = a, \dots, a+k-1$, and $n-k-1$ successive zeros at $m = j, \dots, j+n-k-2$ and a generalized zero at $m = j+n-k-1$, $a+k \leq j \in I^{k+1}$. We first show that $u(j+n-k-1) \neq 0$. Since $u(a) = \dots = u(a+k-1) = 0$, $u = C_{k+1}u_{k+1} + \dots + C_n u_n = \hat{C}_{k+1}\hat{u}_{k+1} + \dots + \hat{C}_n \hat{u}_n$ by (9) and (14). Now, again by (9) and (14), there exists a nontrivial solution of (1) satisfying the boundary conditions $u(a) = \dots = u(a+k-1) = 0$, $u(j) = \dots = u(j+n-k-1) = 0$ if and only if $W(k+1, \dots, n)(j) = \hat{W}(k+1, \dots, n)(j) = 0$. If $u(j+n-k-1) = 0$, $W(k+1, \dots, n)(j) = \hat{W}(k+1, \dots, n)(j) = 0$ which contradicts (12) if $n-k$ is even and contradicts (15) if $n-k$ is odd.

Now, assume $u(j+n-k-1) \neq 0$. We can assume $j > a+k$. For suppose $j = a+k$ and $k \geq 1$. Then u has $n-1$ successive zeros at $m = a, \dots, a+n-2$, and $(-1)^{n-k}u(a+k-1)u(a+n-1) > 0$. But, $u(a+k-1) = 0$, which is a contradiction. If $j = a+k$ and $k = 0$, then u has $n-1$ successive zeros at $m = a, \dots, a+n-2$ and $u(a+n-1) \neq 0$. Then u does not have a generalized zero at $m = a+n-1$, which is again a contradiction.

Also note that $u(a+k) \neq 0$ for if $u(a+k) = 0$, then u has $k+1$ successive zeros at $m = a, \dots, a+k$ and $n-(k+1)$ successive zeros at $m = j, \dots, j+n-(k+1)-1$, $a+k+1 \leq j$. Then, as above $W(k+2, \dots, n)(j) = 0$ or $\hat{W}(k+2, \dots, n)(j) = 0$ which contradicts (12) or (15) depending on whether $n-(k+1)$ is even or odd. So, by (9) and (14), $C_{k+1} \neq 0$ and $\hat{C}_{k+1} \neq 0$. Let $n-k$ be even and assume without loss of generality that $C_{k+1} = 1$. Then

$$W(k+1, \dots, n)(j-1) = W(u, u_{k+2}, \dots, u_n)(j-1) = u(j-1)W(k+2, \dots, n)(j)$$

and

$$\begin{aligned} W(k+1, \dots, n)(j) &= W(u, u_{k+2}, \dots, u_n)(j) \\ &= (-1)^{n-k-1}u(j+n-k-1)W(k+2, \dots, n)(j). \end{aligned}$$

Thus, $(-1)^{n-k-1}u(j-1)u(j+n-k-1) > 0$. But u has a generalized zero at $j+n-k-1$ and so, $(-1)^{n-k}u(j-1)u(j+n-k-1) > 0$. Thus, $n-k$ is not even.

By a similar argument, assuming $\hat{C}_{k+1} = 1$, it follows that $n-k$ is not odd. Thus, $Pu(m) + r(m)u(m) = 0$ is disconjugate on I^n and the proof is complete.

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