COMPACT HOMOMORPHISMS OF C^* -ALGEBRAS

F. GHAHRAMANI

(Communicated by John B. Conway)

ABSTRACT. Suppose A is a C^* -algebra and B is a Banach algebra such that it can be continuously imbedded in $B(H)$, the Banach algebra of bounded linear operators on some Hilbert space H . It is shown that if θ is a compact algebra homomorphism from A into B , then θ is a finite rank operator, and the range of θ is spanned by a finite number of idempotents. If, moreover, B is commutative, then θ has the form $\theta(x) = \chi_1(x)E_1 + \cdots + \chi_k(x)E_k$, where $E_1, \dots, E_k$ are fixed mutually orthogonal idempotents in B and $\chi_1, \dots, \chi_k$ are fixed multiplicative linear functionals on A .

Introduction. Suppose A is a commutative, semi-simple, unital Banach algebra. In [8] H. Kamowitz proved that if θ is a compact endomorphism on A , A' is the set of all multiplicative linear functionals on A , and θ^* is the adjoint of θ , then $\bigcap \theta^{*n}(A')$ is finite. One consequence of this result is a characterization of compact endomorphisms of $C(X)$ [8, Corollary 2.2]. In this paper we characterize compact homomorphisms in a more general setting where they are defined from a C^* -algebra into a Banach algebra that has a continuous imbedding in $B(H)$.

The existence of a nonzero compact endomorphism on a Banach algebra implies the existence of a nonzero proper closed two-sided ideal in that algebra, as S. Grabiner has shown in [4].

Throughout this paper "homomorphism" will mean an "algebra homomorphism." We call a set of idempotents $\{e_i : i \in I\}$ mutually orthogonal if $e_i e_j = 0$, whenever $i \neq j$.

To prove our main result we will need the following extension lemma.

LEMMA. *Let A be a C^* -algebra without identity, and let θ be a compact homomorphism from A into a Banach algebra B . Then, there exists a compact homomorphism $\tilde{\theta}$ from the C^* -unitization $\mathbb{C} \oplus A$ of A into B which extends θ .*

PROOF. Let (e_α) be a bounded approximate identity of A with $\sup_\alpha \|e_\alpha\| \leq 1$, [2, Theorem 12.4]. By the compactness of θ , there exists a subnet $(\theta(e_{\alpha_i}))$ of $(\theta(e_\alpha))$ and an element $e \in B$, such that $\theta(e_{\alpha_i}) \rightarrow e$, in norm. Then for every $x \in A$, we have

$$(1) \quad \begin{aligned} e\theta(x) &= \lim \theta(e_{\alpha_i})\theta(x) = \lim \theta(e_{\alpha_i}x) = \theta(x), \\ \theta(x)e &= \lim \theta(x)\theta(e_{\alpha_i}) = \lim \theta(xe_{\alpha_i}) = \theta(x). \end{aligned}$$

Received by the editors November 6, 1986 and, in revised form, March 9, 1987.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 46K05, 46L05, 47B05; Secondary 43A65, 43A75.

Key words and phrases. Compact homomorphism, finite rank operator, group of unitary elements, semisimple Banach algebra, simple pole, spectral idempotent.

©1988 American Mathematical Society
0002-9939/88 \$1.00 + \$.25 per page

From (1) it follows that

$$(2) \quad \theta(x)e = e\theta(x) = \theta(x)$$

for every $x \in A$. Now if (e_{β_j}) is another subnet with $\theta(e_{\beta_j}) \rightarrow f$ for some f in B , then, as above, we have

$$(3) \quad f\theta(x) = \theta(x)f = \theta(x)$$

for every $x \in A$. Now if in (2) we let $x = e_{\beta_j}$ and in (3) we let $x = e_{\alpha_i}$, and we find limits, we obtain $f = fe = e$. Thus $(\theta(e_{\alpha_i}))$ converges to e . Also, letting $x = e_{\alpha}$ in (2) and finding limits yields $e^2 = e$. Now, we define $\bar{\theta}$ from $\mathbf{C} \oplus A$ into B by $\bar{\theta}(\lambda + x) = \lambda e + \bar{\theta}$. Then from (2) and $e^2 = e$ it follows that $\bar{\theta}$ is a homomorphism. To show that $\bar{\theta}$ is compact, let $\|\cdot\|$ be the norm of the C^* -unitization of A , and let $\|\lambda + x\| \leq 1$. Then by the definition of $\|\cdot\|$ we have $\sup\{\|(\lambda + x)y\| : \|y\| \leq 1\} \leq 1$. In particular, for every e_{α} we have $\|(\lambda + x)e_{\alpha}\| \leq 1$. Now, by the definition of $\bar{\theta}$, and (2), we have

$$\bar{\theta}(\lambda + x) = \lim \theta[(\lambda + x)e_{\alpha}].$$

Thus, the image of the unit ball of $\mathbf{C} \oplus A$ under $\bar{\theta}$ is contained in the closure of the image of the unit ball of A under θ , and the compactness of $\bar{\theta}$ follows from the compactness of θ .

THEOREM 1. *Let A be a C^* -algebra and B be a Banach algebra such that it can be continuously imbedded in $B(H)$. If θ is a compact homomorphism from A into B , then θ is a finite rank operator, and the range of θ is spanned by a finite number of idempotents.*

PROOF. By the lemma we may (and do) assume that A has a unit 1. The case $\theta = 0$ being trivial, we assume $\theta \neq 0$. Now let φ be the continuous imbedding of B in $B(H)$ and let $\mathcal{U}$ be the group of unitary elements (i.e., elements x , with $xx^* = x^*x = 1$) of A . Then by the compactness of $\varphi\theta$, the norm closure $G = [\varphi\theta(\mathcal{U})]^-$ is compact. We prove that G is a compact topological group. To this end, let x and y belong to G . Then $x = \lim \varphi\theta(u_n)$ and $y = \lim \varphi\theta(v_n)$ with the u_n 's and the v_n 's unitary. Then $xy = \lim \varphi\theta(u_n)\varphi\theta(v_n) = \lim \varphi\theta(u_nv_n) \in G$. Now joint continuity of product in Banach algebras shows that G is a compact topological semigroup. Next we prove that G is a group. Let $x \in G$, and let (u_n) be a sequence of unitary elements with $\varphi\theta(u_n) \rightarrow x$. Then by the compactness of $\varphi\theta$, the sequence $\varphi\theta(u_n^*)$ has a convergent subsequence, $(\varphi\theta(u_{n_k}^*))$ say, with $\lim \varphi\theta(u_{n_k}^*) = y \in B$. Then,

$$\varphi\theta(1) = \lim \varphi\theta(u_{n_k}^* u_{n_k}) = \lim \varphi\theta(u_{n_k}^*) \varphi\theta(u_{n_k}) = yx,$$

$$\varphi\theta(1) = \lim \varphi\theta(u_{n_k} u_{n_k}^*) = \lim \varphi\theta(u_{n_k}) \varphi\theta(u_{n_k}^*) = xy.$$

Thus, G is a group with $\varphi\theta(1)$ as its identity. Since G is bounded in $B(H)$, the map $x \mapsto x^{-1}$ ($x \in G$) is continuous. Thus, G is a compact topological group. Further we lose no generality if we assume that $\varphi\theta(1) = I$, the identity operator on H , for, if $\varphi\theta(1) = p$, then $p^2 = p$, and p is the identity operator on the Hilbert space $p(H)$. Then $x \mapsto \varphi \circ \theta(x)|_{p(H)}$ will be a compact imbedding of A into the algebra of bounded linear operators on $p(H)$. Following [5, 22.23, p. 351] we now define a new inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on H by

$$(1) \quad \langle\langle \xi, \eta \rangle\rangle = \int_G \langle g\xi, g\eta \rangle dg,$$

where $\langle \cdot, \cdot \rangle$ is the original inner product on H , $\int_G \cdots dg$ is the normalized Haar integration on G , and where $g\xi$ stands for the image of the vector $\xi \in H$ under the operator $g \in G$. Then H with the new inner product is a Hilbert space, on which G acts as a group of unitary operators [5, 22.23]. Let K denote the vector space H with the new inner product. From (1) and the boundedness of G in $B(H)$ and the open mapping theorem it follows that H and K are isomorphic as Banach spaces. Hence the identity map i from $B(H)$ onto $B(K)$ is continuous. Thus the map $\psi = i \circ \varphi \circ \theta$ is a compact homomorphism from A into $B(K)$, which maps every unitary element in A onto a unitary operator on K . Now if u is a unitary element in A , since both $\psi(u)$ and $\psi(u^*)$ act as unitary operators on K , from

$$I = \psi(1) = \psi(uu^*) = \psi(u)\psi(u^*),$$

it follows that $\psi(u^*) = [\psi(u)]^*$. Since every element in A can be expressed as a linear combination of unitary elements [7, 4.1.7 Theorem], we conclude that $\psi(x^*) = [\psi(x)]^*$, for every $x \in A$. Thus ψ has a closed range [7, 4.1.9 Theorem], and, being a compact operator, it is of finite rank. This implies that θ is of finite rank.

To show that $\theta(A)$ is spanned by a finite number of idempotents, since we have assumed $\theta \neq 0$, we have $\theta(1) \neq 0$, and $\theta(1)$ acts as the identity for the Banach algebra $\theta(A)$. Then there exists a Banach algebra norm $\|\cdot\|'$ on $\theta(A)$, equivalent to the original norm of $\theta(A)$, with $\|\theta(1)\|' = 1$, [1, p. 19]. So we may (and do) assume that $\|\theta(1)\| = 1$. Now let u be a unitary element in A , then by the compactness of θ , the closure of $\{\theta(u^n) : n = 1, 2, \dots\}$ is a compact semigroup. Thus, by [6, Theorem 4] the spectrum of $\theta(u)$ is contained in the closed unit disk, and, its intersection with the unit circle is either empty or consists of a finite number of simple poles. Since this also holds for $\theta(u^{-1}) = [\theta(u)]^{-1}$, and $\text{sp}((\theta(u))^{-1}) = \{1/z : z \in \text{sp}(\theta(u))\}$, it follows that $\text{sp}(\theta(u))$ consists only of a finite number of simple poles, $\lambda_1, \lambda_2, \dots, \lambda_n$ say, located on the unit circle. Now let e_i be the spectral idempotent corresponding to λ_i ($i = 1, 2, \dots, n$). Then $e_i e_j = 0$ ($i \neq j$) and

$$(2) \quad e_1 + \cdots + e_n = I = \theta(1).$$

Then since λ_i 's are simple poles for $\theta(u)$, by [3, Theorem VII.3.18], we have

$$(3) \quad [\theta(u) - \lambda_i]e_i = 0 \quad (i = 1, 2, \dots, n),$$

where we have identified element a of $\theta(A)$ with the operator $\rho_a : b \mapsto ba$ ($b \in \theta(A)$). Now if we multiply the two sides of (2) by $\theta(u)$ and use (3), we obtain

$$(4) \quad \theta(u) = \lambda_1 e_1 + \cdots + \lambda_n e_n.$$

Since A is spanned by the set of all unitary elements, and $\theta(A)$ is finite dimensional, there exists a finite number of unitaries $u_1, \dots, u_n$, such that $\theta(u_1), \dots, \theta(u_n)$ span $\theta(A)$. From this and (4) it follows that $\theta(A)$ is spanned by a finite number of idempotents, and the proof is complete.

From the second part of the proof of Theorem 2, the following is immediate.

COROLLARY 1. *Suppose A is a C^* -algebra and B is a Banach algebra with only a finite number of idempotents. Then a compact homomorphism from A into B is a finite rank operator.*

THEOREM 2. *Suppose A is a C^* -algebra and B is a commutative Banach algebra such that it can be continuously imbedded in $B(H)$, for some Hilbert space H .*

Then a compact homomorphism from A into B is of the form $\theta(x) = \chi_1(x)E_1 + \cdots + \chi_k(x)E_k$, where $\chi_1, \dots, \chi_k$ are multiplicative linear functionals on A and $E_1, \dots, E_k$ are mutually orthogonal idempotents.

PROOF. By the proof of Theorem 1 there exists a finite number of unitary elements $u_1, \dots, u_l$ in A , such that $\theta(u_1), \dots, \theta(u_l)$ span $\theta(A)$. Now suppose

$$\begin{aligned}
 (1) \quad & \theta(u_1) = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n, \\
 & e_i e_j = 0 \ (i \neq j), \quad e_i^2 = e_i, \quad |\alpha_i| = 1 \ (i, j = 1, 2, \dots, n), \\
 & e_1 + \cdots + e_n = I, \\
 & \theta(u_2) = \beta_1 f_1 + \cdots + \beta_m f_m, \\
 & f_i f_j = 0 \ (i \neq j), \quad f_i^2 = f_i, \quad |\beta_i| = 1 \ (i, j = 1, 2, \dots, m), \\
 & f_1 + \cdots + f_m = I.
 \end{aligned}$$

Then $\theta(u_1)$ and $\theta(u_2)$ can be rewritten as

$$\begin{aligned}
 \theta(u_1) &= \alpha_1 e_1 (f_1 + \cdots + f_m) + \cdots + \alpha_n e_n (f_1 + \cdots + f_m), \\
 \theta(u_2) &= \beta_1 (e_1 + \cdots + e_n) f_1 + \cdots + \beta_m (e_1 + \cdots + e_n) f_m.
 \end{aligned}$$

Thus $\theta(u_1)$ and $\theta(u_2)$ can be expressed as linear combinations of $e_i f_j$'s ($i = 1, \dots, n; j = 1, \dots, m$) which are mutually orthogonal idempotents, by the commutativity of B . Repeating this procedure for other $\theta(u_i)$'s we find a finite number of mutually orthogonal idempotents $E_1, \dots, E_k$ such that each $\theta(u_i)$ ($i = 1, \dots, l$) can be expressed as a linear combination of $E_1, E_2, \dots, E_k$, and, since $\theta(u_1), \dots, \theta(u_l)$ span $\theta(A)$, for every $x \in A$, we will then have

$$\theta(x) = \chi_1(x)E_1 + \chi_2(x)E_2 + \cdots + \chi_k(x)E_k.$$

From $\theta(xy) = \theta(x)\theta(y)$ and the fact that $E_1, \dots, E_k$ are mutually orthogonal idempotents, it follows that $x \mapsto \chi_i(x)$ ($i = 1, \dots, k$) defines a multiplicative linear functional on A , and the proof is complete.

COROLLARY 2. Suppose A is a C^* -algebra and B is a Banach algebra of one of the following types:

- (a) commutative semisimple algebras,
- (b) A^* -algebras, in particular, C^* -algebras and the measure algebras $M(G)$ and the group algebras $L^1(G)$ of locally compact groups,
- (c) the algebras $L^p(G)$, $1 \leq p \leq \infty$, of compact groups.

Then a compact homomorphism from A into B is a finite rank operator, and its range is spanned by idempotents.

PROOF. If B is a commutative semisimple Banach algebra, then it has a continuous imbedding in $C_0(X)$ for some compact Hausdorff space X , by the Gelfand transform. Being a C^* -algebra $C_0(X)$ has a faithful representation on some Hilbert space. Also it is well known that each of the Banach algebra in (b)–(c) has a continuous imbedding in $B(H)$. The result now follows from Theorem 1.

ACKNOWLEDGMENTS. The author is grateful to Professor Anthony T. Lau and the Department of Mathematics at the University of Alberta for their hospitality.

REFERENCES

1. F. F. Bansall and J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin, 1973.
2. R. S. Doran and J. Wichman, *Approximate identities and factorization in Banach modules*, Lecture Notes in Math., vol. 768, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.
3. N. Dunford and J. T. Schwartz, *Linear operators*. I, Wiley, New York, 1958.
4. S. Grabiner, *Compact endomorphisms and closed ideals in Banach algebras*, Proc. Amer. Math. Soc. **224** (1984), 547–548.
5. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. I, Springer-Verlag, Berlin, 1963.
6. M. A. Kaashoek and T. T. West, *Locally compact monothetic semi-algebras*, Proc. London Math. Soc. **18** (1968), 428–438.
7. R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, vol. I, Academic Press, New York and London, 1983.
8. H. Kamowitz, *Compact endomorphisms of Banach algebras*, Pacific J. Math. **892** (1980), 313–325.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA,
CANADA T6G2G1

Current address: Department of Mathematics and Astronomy, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2