MARTIN COMPACTIFICATIONS OF THE PUNCTURED DISK WITH CLOSE TO ROTATION FREE DENSITIES

TOSHIMASA TADA

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ABSTRACT. The purpose of this paper is to prove that Martin compactifications Ω_P^* and Ω_Q^* of the punctured unit disk $\Omega: 0 < |z| < 1$ with respect to equations $\Delta u = Pu$ and $\Delta u = Qu$, respectively, are homeomorphic to each other if $|P(z) - Q(z)| = O(|z|^{-2})$ ($z \rightarrow 0$) and $P(z) = P(|z|)$ ($z \in \Omega$).

Before stating our result more precisely we start fixing terminologies. We denote by Ω the punctured unit disk $0 < |z| < 1$ which is viewed as an end of the punctured sphere $0 < |z| \leq \infty$ so that the unit circle $|z| = 1$ is the relative boundary $\partial\Omega$ of Ω and the origin $z = 0$ is the ideal boundary $\delta\Omega$ of Ω . By a *density* P on Ω we mean a nonnegative locally Hölder continuous function $P(z)$ on $\bar{\Omega} = \Omega \cup \partial\Omega$. For a density P on Ω we consider the Martin compactification Ω_P^* of Ω with respect to the equation

$$L_P u \equiv \Delta u - Pu = 0 \quad (\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2)$$

on Ω ([3], cf. also [6, 1]). In the case that P is a *rotation free* density on Ω , i.e. a density P on Ω satisfying $P(z) = P(|z|)$ ($z \in \Omega$), the Martin compactification Ω_P^* is characterized completely by Nakai [4] in terms of a *singularity index* $\alpha(P)$ of P at $\delta\Omega$:

$$\Omega_P^* \cong \{\alpha(P) \leq |z| \leq 1\},$$

i.e. the homeomorphism π_P from Ω to $\alpha(P) < |z| < 1$ defined by

$$\pi_P(z) = (\alpha(P) + (1 - \alpha(P))|z|)z/|z|$$

can be extended to a homeomorphism π_P^* from Ω_P^* to $\alpha(P) \leq |z| \leq 1$ and every point in the ideal boundary $\Omega_P^* - \Omega$ of Ω is minimal. Here $\alpha(P)$ is a value in $[0, 1)$ depending on the singular behavior of P at $\delta\Omega$ (cf. §1) and the only relevant fact about $\alpha(P)$ for the Martin compactification is whether $\alpha(P) = 0$ or not. For example, $\alpha(P)$ of the density $P(z) = |z|^s$ on Ω satisfies [4, 5]

$$\alpha(P) = 0 \quad \text{if and only if} \quad -2 \leq s \leq \infty,$$

and hence

$$\Omega_P^* \cong \{|z| \leq 1\} \quad \text{if} \quad -2 \leq s \leq \infty.$$

This fact is partly extended to Ω_P^* with a general density P on Ω by Kawamura [2]: a general density P on Ω satisfies

$$(1) \quad \Omega_P^* \cong \{|z| \leq 1\} \quad \text{if} \quad P(z) = O(|z|^{-2}) \quad \text{as} \quad z \rightarrow 0.$$

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In this paper we show that Ω_Q^* with a general density Q on Ω is homeomorphic to Ω_P^* with a rotation free density P on Ω if Q is close to P .

THEOREM. *Let P be a rotation free density on Ω and Q be a general density on Ω with*

$$|P(z) - Q(z)| = O(|z|^{-2}) \quad (z \rightarrow 0),$$

then $\Omega_P^ \cong \Omega_Q^*$, i.e. the identity mapping I_Ω of Ω is extended to a homeomorphism I_Ω^* from Ω_P^* to Ω_Q^* and every point in $\Omega_Q^* - \Omega$ is minimal.*

From this theorem (1) follows since $\alpha(P) = 0$ if $P \equiv 0$.

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1. Singularity indices. We recall the definition and fundamental properties of singularity indices of rotation free densities on Ω according to [4].

Let P be a rotation free density on Ω . The unique bounded solution e_P of $L_P u = 0$ on Ω with boundary values 1 on $\partial\Omega$ is referred to as the P -unit on Ω . We also consider rotation free densities P_n ($n = 0, 1, \dots$) on Ω defined by $P_n(z) = P(z) + n^2|z|^{-2}$ ($z \in \Omega$) and denote by e_n the P_n -unit on Ω , where we follow the convention $P_0 = P$ and $e_0 = e_P$. Then e_n is positive on Ω and rotation free, i.e. $e_n(z) = e_n(|z|)$ ($z \in \Omega$). Each function $e_n(r)/e_0(r)$ ($n = 1, 2, \dots$) of r in $(0, 1]$ is increasing so that there exists the limit

$$\alpha_n(P) = \lim_{r \rightarrow 0} \frac{e_n(r)}{e_0(r)}$$

which is referred to as the n th singularity index of P at $\delta\Omega$. In particular we denote by $\alpha(P)$ the 1st singularity index $\alpha_1(P)$ and call it simply the singularity index of P at $\delta\Omega$. Then we have the following fundamental inequalities [4]

$$(2) \quad 0 \leq \alpha(P)^{(3^n-1)/2} \leq \alpha_n(P) \leq \alpha(P)^n < 1 \quad (n = 1, 2, \dots).$$

2. Comparison between Martin compactifications. For general densities P and Q on Ω with $P \leq Q$, we show that $\Omega_P^* \cong \Omega_Q^*$ if the Q -Green's function is comparable with the P -Green's function.

Consider general densities P and Q on Ω with $P(z) \leq Q(z)$ ($z \in \Omega$). With the density P (Q , resp.) we associate the class $PP(\Omega; \partial\Omega)$ ($QP(\Omega; \partial\Omega)$, resp.) of nonnegative solutions u of $L_P u = 0$ ($L_Q u = 0$, resp.) on Ω with vanishing boundary values on $\partial\Omega$. For any function u in $PP(\Omega; \partial\Omega)$, the sequence $\{v_n\}_2^\infty$ of solutions v_n of $L_Q v = 0$ on $\{1/n < |z| < 1\}$ with boundary values u on $\{|z| = 1/n\} \cup \{|z| = 1\}$ is decreasing. Then $\{v_n\}$ converges to a function Q_u in $QP(\Omega; \partial\Omega)$ uniformly on every compact subset of Ω . Similarly we may construct a function P_v in $PP(\Omega; \partial\Omega) \cup \{\infty\}$ for every v in $QP(\Omega; \partial\Omega)$. We remark that $Q_u \leq u$, $P_v \geq v$, $\lambda Q_u \not\leq u$, and $P_v \not\geq \lambda v$ for any constant λ with $\lambda > 1$ if $Q_u \neq 0$ and $P_v \neq \infty$. Assume that u is a minimal function in $PP(\Omega; \partial\Omega)$ and v is a function in $QP(\Omega; \partial\Omega)$ with $u \geq v > 0$. Set $\lambda_0 = \max\{\lambda: Q_u \geq \lambda v\}$. Then $\lambda_0 \geq 1$ by $Q_u \geq v$. Further $u \geq \lambda_0 v$ implies $u \geq P_{\lambda_0 v} \geq \lambda_0 v$, and hence $P_{\lambda_0 v} = \mu u$ for a constant μ with $0 < \mu \leq 1$. Therefore $u = P_{(\lambda_0/\mu)v} \geq (\lambda_0/\mu)v$ implies $Q_u \geq (\lambda_0/\mu)v$ so that $\mu = 1$ by the definition of λ_0 . Thus $u \geq P_{Q_u} = P_{\lambda_0 v} + P_{Q_u - \lambda_0 v} = u + P_{Q_u - \lambda_0 v}$ and $0 \leq Q_u - \lambda_0 v \leq P_{Q_u - \lambda_0 v}$ yield $Q_u = \lambda_0 v$. Then we have the following lemma.

LEMMA 1. *Let u be a minimal function in $PP(\Omega; \partial\Omega)$. Then*

$$\{v \in QP(\Omega; \partial\Omega) : u \geq v\} = \{\lambda Q_u : 0 \leq \lambda \leq 1\},$$

and hence Q_u is minimal if $Q_u \neq 0$.

We have also the following

LEMMA 2. *Let u_1 and u_2 be minimal functions in $PP(\Omega; \partial\Omega)$. Then $\{v \in QP(\Omega; \partial\Omega) : u_1 \geq v, u_2 \geq v\} = \{0\}$ if u_1 is nonproportional to u_2 .*

In fact, if a function v in $QP(\Omega; \partial\Omega) - \{0\}$ is dominated by u_1 and u_2 , then P_v is proportional to u_1 and u_2 .

We denote by G_P (G_Q , resp.) the P -Green's (Q -Green's, resp.) function on Ω , i.e. the Green's function with respect to $L_P u = 0$ ($L_Q u = 0$, resp.). Since P is dominated by Q , G_P dominates G_Q . Assume that there exist a point z_0 in Ω , a positive constant C , and a positive number δ with $\delta < 1$ such that

$$(3) \quad CG_P(z_0, \zeta) \leq G_Q(z_0, \zeta) \quad (0 < |\zeta| < \delta).$$

Then the P -Martin kernel $K_P(z, \zeta) = G_P(z, \zeta)/G_P(z_0, \zeta)$ on Ω and the Q -Martin kernel $K_Q(z, \zeta) = G_Q(z, \zeta)/G_Q(z_0, \zeta)$ on Ω satisfy

$$(4) \quad K_P(z, \zeta) \geq CK_Q(z, \zeta) \quad (z \in \Omega, 0 < |\zeta| < \delta).$$

Let Γ_P^* (Γ_Q^* , resp.) be the set of points in $\Omega_P^* - \Omega$ ($\Omega_Q^* - \Omega$, resp.) over $\delta\Omega$ and ζ^* be any point in Γ_P^* . Take a sequence $\{\zeta_n\}_1^\infty$ of points ζ_n in Ω with $\lim \zeta_n = \zeta^*$ in Ω_P^* . Since $\lim |\zeta_n| = 0$, a subsequence of $\{\zeta_n\}$ converges to a point ξ^* in Γ_Q^* . In view of (4) we have $K_P(z, \zeta^*) \geq CK_Q(z, \xi^*)$ so that by Lemma 1 ξ^* is independent of the choice of the sequence $\{\zeta_n\}$ and its subsequence when ζ^* is minimal. Now assume that every point in Γ_P^* is minimal. Then we may extend the identity mapping I_Ω of Ω to a mapping I_Ω^* from Ω_P^* to Ω_Q^* . We remark that every point in $\Omega_P^* - (\Omega \cup \Gamma_P^*)$ ($\Omega_Q^* - (\Omega \cup \Gamma_Q^*)$, resp.) is minimal and a neighborhood of $\Omega_P^* - (\Omega \cup \Gamma_P^*)$ ($\Omega_Q^* - (\Omega \cup \Gamma_Q^*)$, resp.) is homeomorphic to a neighborhood of $\{|z| = 1\}$ in $\{|z| \leq 1\}$ since P (Q , resp.) is Hölder continuous on $\bar{\Omega}$ (cf. e.g. [1, 4]). Clearly I_Ω^* is surjective. Further I_Ω^* is injective by Lemma 2. Thus we deduce the following

LEMMA 3. *If any point in Γ_P^* is minimal and if (3) is valid, then $\Omega_P^* \cong \Omega_Q^*$, i.e. I_Ω can be extended to a homeomorphism from Ω_P^* to Ω_Q^* and any point in Γ_Q^* is minimal.*

3. Proof of the theorem. Let P be a rotation free density on Ω and Q be a general density on Ω with $|P(z) - Q(z)| = O(|z|^{-2})$ ($z \rightarrow 0$). There exists a positive integer k such that the rotation free density R on Ω given by $R(z) = \max(P(z) - k^2|z|^{-2}, 0)$ ($z \in \Omega$) satisfies

$$\begin{aligned} R(z) &\leq P(z) \leq R(z) + k^2|z|^{-2}, \\ R(z) &\leq Q(z) \leq R(z) + (2k)^2|z|^{-2}. \end{aligned}$$

We assume that $\alpha(R) > 0$ since we have seen [7] that $\alpha(R) = 0$ implies $\Omega_P^* \cong \{|z| \leq 1\}$ and $\Omega_Q^* \cong \{|z| \leq 1\}$. We denote by f_n ($n = 0, 1, \dots$) then R_n -unit and set

$$\begin{aligned} M_R &= \max\{G_R(\tfrac{1}{2}, \zeta) : |\zeta| = 1/4\}, \\ m_P &= \min\{G_P(\tfrac{1}{2}, \zeta) : |\zeta| = 1/4\}. \end{aligned}$$

Then we have

$$\begin{aligned} G_R(\tfrac{1}{2}, \varsigma)/M_R &\leq f_0(\varsigma)/f_0(\tfrac{1}{4}) & (0 < |\varsigma| < 1/4), \\ G_P(\tfrac{1}{2}, \varsigma)/m_P &\geq f_k(\varsigma)/f_k(\tfrac{1}{4}) & (0 < |\varsigma| < 1/4). \end{aligned}$$

Further we have $\alpha_k(R)f_0(\varsigma) \leq f_k(\varsigma)$ ($\varsigma \in \Omega$) and $\alpha_k(R) > 0$ by (2). Therefore the positive constant

$$C_P = m_P f_0(\tfrac{1}{4}) \alpha_k(R) / M_R f_k(\tfrac{1}{4})$$

satisfies

$$C_P G_R(\tfrac{1}{2}, \varsigma) \leq G_P(\tfrac{1}{2}, \varsigma) \quad (0 < |\varsigma| < 1/4).$$

Similarly an inequality

$$C_Q G_R(\tfrac{1}{2}, \varsigma) \leq G_Q(\tfrac{1}{2}, \varsigma) \quad (0 < |\varsigma| < 1/4)$$

is valid for a positive constant C_Q . Thus Lemma 3 and the fact that any point in $\Omega_R^* - \Omega$ is minimal yield that $\Omega_P^* \cong \Omega_R^*$, $\Omega_Q^* \cong \Omega_R^*$, and hence the theorem is proved.

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DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING, NAGOYA INSTITUTE OF TECHNOLOGY, GOKISO SHOWA, NAGOYA 466, JAPAN

Current address: Department of Mathematics, Daido Institute of Technology, Daido, Minami, Nagoya 457, Japan