CONVEX SETS AND SUBHARMONICITY OF THE DISTANCE FUNCTION

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(Communicated by Irwin Kra)

ABSTRACT. In this paper we show that in order for a proper closed subset F of a plane domain D to be convex it is necessary and sufficient that $\text{dist}(\cdot, F)$ be subharmonic in D . We also show that this result does not hold for higher dimensions.

1. Introduction and main results. Throughout this paper F is a nonempty closed subset of $\mathbb{R}^N$ ($N \geq 2$). By CF we denote the complement of F in $\mathbb{R}^N$ and by ∂F the finite boundary of F .

We define the *distance function* d_F from F by

$$d_F(x) = \text{dist}(x, F) = \inf_{y \in F} \|x - y\| \quad (x \in \mathbb{R}^N)$$

where $\|\cdot\|$ denotes the Euclidean norm.

For a given x in $\mathbb{R}^N$ and a positive real number r , $S(x, r)$ denotes the sphere centre x , radius r , and $B(x, r)$ denotes the open ball of centre x and radius r . We say that $B(x, r)$ *touches* F if $B(x, r) \cap F$ is empty and $S(x, r) \cap F$ is nonempty. Further if $B(x, r)$ touches F and $y \in S(x, r) \cap F$ then we say that $B(x, r)$ *touches* F at y . For a function f integrable with respect to the surface area measure σ on $S(x, r)$, $M(f, x, r)$ denotes the surface mean of f over $S(x, r)$.

Given two points, x and y , in $\mathbb{R}^N$ we define $\overline{xy}$ to be the (closed) line segment joining x and y , and xy to be $\overline{xy} \setminus \{x, y\}$. If T is a closed convex set then we call $t \in T$ an *extreme point* of T if t is not an element of $t_1 t_2$ for all points t_1 and t_2 in T .

Motzkin's Theorem [3, Theorem 7.8] states that a nonempty closed set F in $\mathbb{R}^N$ is convex if and only if every point in $\mathbb{R}^N$ has a unique nearest point in F . Armitage and Kuran [1, Theorem 3] used this result to prove that d_F is subharmonic in $\mathbb{R}^N$ if and only if F is convex. In this paper we prove a local Motzkin-type theorem in order to obtain a local version of Armitage and Kuran's result in the case where $N = 2$. We also show that our second result does not hold in higher dimensions.

The results are as follows.

THEOREM 1. *Let F be a nonempty proper closed subset of $\mathbb{R}^N$ ($N \geq 2$), let t be a point of ∂F and let ε be a (strictly) positive real number.*

If every point in $B(t, \varepsilon)$ has a unique nearest point in F then there exists a ball of radius ε which touches F at t .

THEOREM 2. *Let F be a nonempty proper closed subset of $\mathbb{R}^2$ and let D be a domain such that $F \subset D$. Then d_F is subharmonic in D if and only if F is convex.*

Received by the editors March 16, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 31A05; Secondary 52A10.

Key words and phrases. Convex set, subharmonic function.

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0002-9939/88 \$1.00 + \$.25 per page

2. Some preparatory results. We first give a fairly classical result which we do not want to call our own.

LEMMA A. *Let F be closed in $\mathbb{R}^N$ and let $B(x, r)$ be a ball touching F at only one point y_0 . Choose any y in $\overline{B(x, r)} \setminus \{y_0\}$. Then there exists a (strictly) positive number δ such that*

$$(*) \quad \overline{B(x + \varepsilon(y - y_0), r)} \subset CF$$

for all ε in $(0, \delta)$.

For a sketch of the proof we note that if the lemma failed there would exist a null sequence (ε_n) such that $(*)$ did not hold for each ε_n . This would give rise to a convergent sequence of points (x_n) such that $x_n \in \overline{B(x + \varepsilon_n(y - y_0), r)} \cap F$ for each n . The limit x_0 of (x_n) would be distinct from y_0 and would lie in F and on $S(x, r)$ which is not possible since y_0 is the only point at which $B(x, r)$ touches F .

From Armitage and Kuran [1, Lemma 2] we deduce the following whose proof is left to the reader.

LEMMA B. *Let U and F be open and closed respectively in $\mathbb{R}^N$. If d_F is subharmonic in U then every point in U has a unique nearest point in F .*

In the absence of any reference we shall prove the following simple result.

LEMMA 1. *Let F be a closed set in $\mathbb{R}^N$ and let S be a set in $\mathbb{R}^N$ having the property that every point x in S has a unique nearest point $f(x)$ in F . Then $f: S \rightarrow F$ is continuous in S .*

The proof is as follows. Take any sequence (x_n) in S such that (x_n) converges to a point x_0 in S and $(f(x_n))$ converges to a point y_0 in F . It is enough to prove that $y_0 = f(x_0)$. Now

$$\|x_0 - y_0\| = \lim_{n \rightarrow \infty} \|x_n - f(x_n)\| = \lim_{n \rightarrow \infty} d_F(x_n) = d_F(x_0),$$

so by the uniqueness of $f(x_0)$ we obtain the result.

We now move on to the proofs of the theorems.

3. Proof of Theorem 1. Suppose by contradiction that there is no ball of radius ε touching F at t . Hence $d_F < \varepsilon$ in $\overline{B(t, \varepsilon)}$ and so, by the continuity of d_F , there exists a positive real number δ such that $d_F < \delta < \varepsilon$ in $\overline{B(t, \varepsilon)}$. Choose any $q \in B(t, \varepsilon - \delta) \cap CF$ and let Q be the set given by

$$Q = \{x \in \mathbb{R}^N: q \in \overline{B(x, d_F(x))}\} = \{x \in \mathbb{R}^N: d_F(x) \geq \|x - q\|\}.$$

Clearly $Q \subset B(q, \delta) \subset B(t, \varepsilon)$, Q is nonempty and Q is compact. Therefore d_F attains its supremum in Q at a point x_0 of Q . By the hypothesis of the theorem x_0 has a unique nearest point in F so $B(x_0, d_F(x_0))$ touches F at only one point, p say. By Lemma A there exists an α in $(0, 1)$ such that

$$(**) \quad \overline{B(x_0 + \alpha(q - p), d_F(x_0))} \subset CF.$$

Since $\alpha \in (0, 1)$ we have that $q \in B(x_0 + \alpha(q - p), d_F(x_0))$ and therefore $x_0 + \alpha(q - p) \in Q$. By $(**)$ we see that

$$d_F(x_0 + \alpha(q - p)) > d_F(x_0) = \sup_{x \in Q} d_F(x),$$

a contradiction.

4. Proof of Theorem 2. The “if” part of the theorem is covered by [1, Theorem 3] so it remains to prove the “only if” part.

By Lemma B every point in D has a unique nearest point in F so the function f of Lemma 1 is continuous in D . Since D is a domain and $f(D) = F$ we have that F is path connected. Suppose, by contradiction, that F is nonconvex. By [3, Theorem 4.2] there exist points x and y in F such that $xy \subset CF$. The path connectedness of F implies that there exists a path in F with no self-intersections joining x and y (see [4, 31.6, p. 222]). By composing this path with $\overline{xy}$ we obtain a Jordan curve γ and by the Jordan Curve Theorem γ is the boundary of a bounded domain $\text{int } \gamma$. Now let Ω be the open set equal to $CF \cap \text{int } \gamma$. Since $xy \subset CF$ it follows that $\overline{xy} \subset \partial\Omega$, therefore $\overline{xy}$ is contained in the convex hull T of $\overline{\Omega}$ and consequently no point of xy is an extreme point of T . The Krein-Milman Theorem (see [2, p. 1]) says that T is the convex hull of its extreme points, so T has at least 3 extreme points and hence T has an extreme point which does not lie on $\overline{xy}$. Without loss of generality this point may be taken as the origin 0 and we may suppose (since T is convex) that $T \subset \overline{\Omega}_0$ where $\Omega_0 = \{(\xi, \eta) \in \mathbb{R}^2: \eta > 0\}$. Since 0 is an extreme point of T it follows that $0 \in \partial\Omega$. Now $d_{\overline{xy}}(0) > 0$ so $0 \in F$ and therefore $d_{CD}(0) > 0$, hence we may suppose without loss of generality that $\min\{d_{CD}(0), d_{\overline{xy}}(0)/2\} = 1$. Now

$$d_{C\Omega} = \min\{d_\gamma, d_F\} = \min\{d_{\overline{xy}}, d_F\}$$

in Ω , so $d_{C\Omega} = d_F$ in $B(0, 1) \cap \Omega$ and hence $d_{C\Omega}$ is subharmonic in $B(0, 1)$. By Lemma B every point in $B(0, 1)$ has a unique nearest point in $C\Omega$, so by Theorem 1, $B((0, 1), 1) \subset \Omega$. Now let L_1 and L_2 be the sets $\{(\xi, 0): \xi > 0\}$ and $\{(\xi, 0): \xi < 0\}$ respectively. The extremity of 0 with respect to T implies that either $L_1 \cap T$ or $L_2 \cap T$ is empty. Without loss of generality we may suppose that $L_1 \cap T$ is empty. This gives that $d_{C\Omega} < h$ in $\{(\xi, \eta): \xi > 0, \eta > 0\}$ where $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $h((\xi, \eta)) = \eta((\xi, \eta) \in \mathbb{R}^2)$. Further $d_{C\Omega} \leq h$ in $\mathbb{R}^2$ and so

$$\begin{aligned} d_F((0, \tfrac{1}{2})) &= d_{C\Omega}((0, \tfrac{1}{2})) = \tfrac{1}{2} = h((0, \tfrac{1}{2})) = \mathcal{M}(h, (0, \tfrac{1}{2}), \tfrac{1}{4}) \\ &> \mathcal{M}(d_{C\Omega}, (0, \tfrac{1}{2}), \tfrac{1}{4}) = \mathcal{M}(d_F, (0, \tfrac{1}{2}), \tfrac{1}{4}), \end{aligned}$$

a contradiction.

5. A counterexample for higher dimensions. In this section we show that Theorem 2 does not hold for $N \geq 3$.

Let F be the circumference $\{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N: x_1^2 + x_2^2 = 1, x_3 = x_4 = \dots = x_N = 0\}$ and let ρ denote $\sqrt{x_1^2 + x_2^2}$. Then

$$d_F(x) = \sqrt{(\rho - 1)^2 + x_3^2 + \dots + x_N^2}$$

and so

$$\Delta d_F(x) = \frac{N - 1 - 1/\rho}{\sqrt{(\rho - 1)^2 + x_3^2 + \dots + x_N^2}} \quad (\rho \neq 0).$$

Therefore d_F is subharmonic in $D = \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N: x_1^2 + x_2^2 > (N - 1)^{-2}\}$ which contains F . Since F is not convex, we have shown that Theorem 2 fails in higher dimensions.

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