BOUNDED SEQUENCE-TO-FUNCTION HAUSDORFF TRANSFORMATIONS

CONSTANTINE GEORGAKIS

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ABSTRACT. Let

$$(Ta)(y) = \sum_{n=0}^{\infty} (-y)^n \frac{g^{(n)}(y)}{n!} a_n, \quad y \geq 0,$$

be the sequence-to-function Hausdorff transformation generated by the completely monotone function g or, what is equivalent, the Laplace transform of a finite positive measure σ on $[0, \infty)$. It is shown that for $1 \leq p \leq \infty$, T is a bounded transformation of l^p with weight $\Gamma(n+s+1)/n!$ into $L^p[0, \infty)$ with weight y^s , $s > -1$, whose norm $\|T\| = \int_0^\infty t^{-(1+s)/p} d\sigma(t) = C(p, s)$ if and only if $C(p, s) < \infty$, and that for $1 < p < \infty$, $\|Ta\|_{p,s} < C(p, s)\|a\|_{p,s}$ unless a_n is a null sequence. Furthermore, if $1 < p < r < \infty$, $0 < \lambda < 1$, $1/r = 1/p + \lambda - 1$ and σ is absolutely continuous with derivatives ψ such that the function $\psi_r(t) = t^{-1/r}\psi(t)$ belongs to $L^{1/\lambda}[0, \infty)$, then the transformation $(T_\lambda a)(y) = y^{1-\lambda}(Ta)(y)$ is bounded from l^p to $L^r[0, \infty)$ and has norm $\|T_\lambda\| \leq \|\psi_r\|_{1/\lambda}$. The transformation T includes in particular the Borel transform and that of generalized Abel means. These results constitute an improved analogue of a theorem of Hardy concerning the discrete Hausdorff transformation on l^p which corresponds to a totally monotone sequence, and lead to improved forms of some inequalities of Hardy and Littlewood for power series and moment sequences.

Hardy showed that the discrete Hausdorff transformation

$$(Ha)_n = \sum_{k=0}^n \binom{n}{k} (\Delta^{n-k} \mu_k) a_k, \quad 1 < p < \infty,$$

where μ_n is a totally monotone sequence or, equivalently, $\mu_n = \int_0^1 t^n d\sigma(t)$, where σ is a probability measure on $[0, 1]$, such that $\sigma(\{0\}) = 0$ and $C_p = \int_0^1 t^{-1/p} d\sigma(t) < \infty$, is a bounded transformation on l^p and $\|Ha\|_p \leq C_p \|a\|_p$. Moreover, C_p is the best possible constant and exact equality occurs only if a_n is a null sequence or T is the identity transformation [2] or [3].

Our aim here is to establish the analogue of Hardy's Theorem for the Hausdorff transformation generated by a completely monotone function and to strengthen it in the case of the measure σ is an absolutely continuous with derivative in L^p . Then we show how these results lead to certain L^p inequalities for the Borel transform and the transformation of generalized Abel means. These in turn give rise to improved

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versions of some inequalities for power series and moment sequences of Hardy and Littlewood [4].

The Hausdorff transformation generated by a completely monotone function g or, what is equivalent (see [11]), the Laplace transform of a finite positive Borel measure σ on $[0, \infty)$ is given by

$$Ta(y) = \sum_{n=0}^{\infty} (-y)^n \frac{g^{(n)}(y)}{n!} a_n = \sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-yt} \frac{(yt)^n}{n!} d\sigma(t), \quad y \geq 0.$$

The transformation T which maps a sequence a_n to the function $Ta(y)$, $y \geq 0$, is known as the Henriksson transformation or J -transform. It reduces to Borel summation when σ is a unit mass at 1 and to the generalized Abel means of order $\nu > 0$ when σ is the gamma probability density $t^{\nu-1}e^{-t}/\Gamma(\nu)$. The transformation T was introduced by Henriksson [5] as a generalization of Abel and Borel summability, and its summability properties have been studied by Henriksson [6], Jakimovski [7], Leviatan and Lorch [9], Borwein [1], Renyi [10] and others. Theorems 1 and 2 below describe some of the L^p -continuity properties of the transformation T .

In the sequel we denote by $l^{p,s}$ and $L^{p,s}[0, \infty)$ the space of sequences $a = \{a_n\}_{n=0}^{\infty}$ and functions, $f(y)$, $y \geq 0$, respectively, such that

$$\|a\|_{p,s}^p = \sum_{n=0}^{\infty} |a_n|^p (n+1)_s < \infty,$$

$$\|f\|_{p,s}^p = \int_0^{\infty} |f(y)|^p y^s dy < \infty,$$

where $s > -1$, $(n+1)_s = \Gamma(n+s+1)/n!$ is the Pochhammer symbol, and $p' = p/(p-1)$ is the conjugate index of p . We shall also need the following inequalities for the gamma function and the binomial coefficients.

LEMMA 1. (a) If $u > 0$, $v > 0$ then,

$$\begin{aligned} \Gamma(u+v)/\Gamma(u) &\leq u^v \quad \text{for } 0 < v \leq 1, \text{ and} \\ &\geq u^v \quad \text{for } v > 1, \end{aligned}$$

(b) $(n+1)_s \leq (n+1)^{s+1}/(n+s+1)$ for $n \geq 0$, $-1 < s < 0$,

(c) $(n+1)_s \leq (n+s+1)^s$ for $n \geq 0$, $s \geq 0$.

PROOF. (a) is simply a consequence of Jensen's inequality;

$$\left[\frac{\Gamma(u+v)}{\Gamma(u)} \right]^{1/v} = \left[\int_0^{\infty} \frac{t^v t^{u-1} e^{-t} dt}{\Gamma(u)} \right]^{1/v} \leq \text{or } \geq \int_0^{\infty} \frac{t^u e^{-t} dt}{\Gamma(u)} = u$$

depending on $v \leq 1$ or $v > 1$. To prove (b) let $u = n+1$, $v = s+1$ in (a). Then $u > 0$, $0 < v \leq 1$, and

$$(n+1)_s = \frac{\Gamma(n+s+2)}{\Gamma(n+1)(n+s+1)} \leq \frac{(n+1)^{s+1}}{n+s+1}.$$

Finally, to prove (c) let $u = n + [s+1]$, $v = s+1 - [s+1]$ in (a). Then $u > 0$, $0 < v < 1$ since we may assume $s > 0$, and by (a) we have

$$\begin{aligned} n!(n+1)_s(n+[s+1]) &= (n+[s+1])!\Gamma(n+[s+1]+v)/\Gamma(n+[s+1]) \\ &\leq n!(n+[s+1])^{[s+1]}(n+[s+1])^v. \end{aligned}$$

Hence,

$$(n+1)_s \leq (n+[s+1])^s \leq (n+s+1)^s.$$

LEMMA 2. If $C_{n\nu} = \binom{n+\nu-1}{\nu-1}$, n is an integer and ν is real, then we have

- (a) $C_{n\nu} \geq (n+1)^{\nu-1} / \max(\Gamma(\nu), 1)$ for $n \geq 0$, $\nu > 0$,
- (b) $C_{n\nu} \leq (n+1)^{\nu-1} / \min(\Gamma(\nu), 1)$ for $n \geq 0$, $\nu > 1$,
- (c) $C_{n\nu} \leq n^{\nu-1} / \Gamma(\nu)$ for $n \geq 1$, $0 < \nu \leq 1$.

PROOF. In proving (a) and (b) we may assume $n \geq 1$, since both are clearly valid for $n = 0$. For $0 < \nu < 1$, (a) is equivalent to $\Gamma(n+1)/\Gamma(n+\nu) \leq (n+1)^{1-\nu}$ which follows by Lemma 1(a) with $u = n+\nu$, $v = 1-\nu$. For $\nu > 2$, (a) also follows from Lemma 1(a) with $u = n+1$, $v = \nu-1$. To prove (a) for $1 < \nu < 2$, we have

$$\begin{aligned} \left[\frac{\Gamma(n+1)\Gamma(\nu)}{\Gamma(n+\nu)} \right]^{1/(\nu-1)} &= \left[\int_0^1 (1-x)^{\nu-1} x^{n-1} n dx \right]^{1/(\nu-1)} \\ &\leq \int_0^1 (1-x) x^{n-1} n dx = \frac{1}{n+1}, \quad \text{since } \frac{1}{\nu-1} > 1. \end{aligned}$$

The last inequality gets reversed when $1/(\nu-1) < 1$, and establishes (b) for $\nu > 2$. For $1 < \nu < 2$, (b) is equivalent to $\Gamma(n+1)/\Gamma(n+\nu) \leq (n+1)^{\nu-1}$ which is Lemma 1(a) with $u = n+1$, $v = \nu-1$. Finally, for $0 < \nu \leq 1$, (c) follows similarly from the inequality $\Gamma(n+\nu)/\Gamma(n) \leq n^{\nu-1}$ which is Lemma 1(a) with $u = n$, $v = \nu$.

THEOREM 1. Let σ be a complex Borel measure of finite variation on $[0, \infty)$ such that $\sigma(\{0\}) = 0$ and T be the transformation given by

$$(Ta)(y) = \sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-yt} \frac{(yt)^n}{n!} d\sigma(t), \quad y \geq 0,$$

where

$$C(p, s) = \int_0^{\infty} t^{-(1+s)/p} d|\sigma|(t) < \infty, \quad s > -1,$$

and $a = \{a_n\}_{n=0}^{\infty}$ is a sequence in $l^{p,s}$. Then we have

- (a) T is a bounded transformation of $l^{p,s}$ into $L^{p,s}[0, \infty)$ and

$$\|Ta\|_{p,s} \leq C(p, s) \|a\|_{p,s}, \quad \text{i.e., } \|T\| \leq C(p, s) \text{ for } 1 \leq p \leq \infty.$$

- (b) If in addition σ is a positive measure, then (i) $\|Ta\|_{p,s} < C(p, s) \|a\|_{p,s}$ for $1 < p < \infty$ unless the sequence a_n is null; (ii) $\|T\| = C(p, s)$ for $1 \leq p \leq \infty$; (iii) the condition $C(p, s) < \infty$ is also necessary for T to be bounded.

PROOF. For $p = \infty$, (a) is clearly true, and for $p = 1$, it follows by Fubini's theorem. Let $1 < p < \infty$, $a \in l^{p,s}$, and consider the transformation E_t , $t > 0$, given by

$$(E_t a)(y) = \sum_{n=0}^{\infty} e^{-yt} \frac{(yt)^n}{n!} |a_n|, \quad y \geq 0.$$

Then

$$\begin{aligned}(E_t a)(y) &= \sum_{n=0}^{\infty} (e^{-yt} \frac{(yt)^n}{n!} |a_n|)^{1/p} (e^{-yt} \frac{(yt)^n}{n!})^{1/p'} \\ &\leq \left(\sum_{n=0}^{\infty} e^{-yt} \frac{(yt)^n}{n!} |a_n|^p \right)^{1/p}\end{aligned}$$

by Hoelder's inequality. Moreover, if $t > 0$, $y > 0$ and we have exact equality, then

$$|a_n|^p e^{-yt} (yt)^n / n! = K e^{-yt} \frac{(yt)^n}{n!} \quad \text{for } n \geq 0$$

for some positive constant K . This implies that $|a_n|^p = K$ for $n \geq 0$ and that $\|a\|_{p,s}^p = K^p \sum_{n=0}^{\infty} (n+1)_s$, which is divergent for $s > -1$ and leads to contradiction. Therefore,

$$(*) \quad (E_t a)(y)^p < \sum_{n=0}^{\infty} e^{-yt} \frac{(yt)^n}{n!} |a_n|^p$$

unless the sequence a_n is null. Let $f \in L^{p',s}[0, \infty)$. Using the Hoelder's inequality for a double integral and the preceding inequality for $(E_t a)(y)^p$, we get

$$\begin{aligned}|(Ta, f)| &= \left| \int_0^{\infty} (Ta)(y) f(y) y^s dy \right| \\ &\leq \int_0^{\infty} |f(y)| y^s dy \int_0^{\infty} (E_t a)(y) d|\sigma|(t) \\ &\leq \int_0^{\infty} \int_0^{\infty} [(E_t a)(y) t^{(1+s)/pp'}] [|f(y)| t^{-(1+s)/pp'}] y^s dy d|\sigma|(t) \\ &\leq \left[\int_0^{\infty} \int_0^{\infty} (E_t a)(y)^p t^{(1+s)/p'} y^s dy d|\sigma|(t) \right]^{1/p} C(p, s)^{1/p'} \|f\|_{p',s} \\ &\leq \left[\int_0^{\infty} \int_0^{\infty} \left(\sum_{n=0}^{\infty} e^{-yt} \frac{(yt)^n}{n!} |a_n|^p \right) t^{(1+s)/p'} y^s dy d|\sigma|(t) \right]^{1/p} C(p, s)^{1/p'} \|f\|_{p',s} \\ &= C(p, s)^{1/p} \|a\|_{p,s} C(p, s)^{1/p'} \|f\|_{p',s} \\ &= C(p, s) \|a\|_{p,s} \|f\|_{p',s}.\end{aligned}$$

Hence, $\|Ta\|_{p,s} \leq C(p, s) \|a\|_{p,s}$. Furthermore, the preceding argument shows that if σ is positive, $a_n \geq 0$, $f \geq 0$, and $(Ta, f) = C(p, s) \|a\|_{p,s} \|f\|_{p',s}$ then $(*)$ holds a.e. for $y > 0$ and a.e. relative to the measure σ for $t > 0$ and, hence, a_n is a null sequence.

(b) For the rest of the proof we let σ be a finite positive measure.

(i) Suppose $\|Ta\|_{p,s} = C(p, s) \|a\|_{p,s}$ for some sequence a_n in $l^{p,s}$. Then $a_n \geq 0$, because $\|Ta\|_{p,s} \leq \|T|a|\|_{p,s}$, where $|a| = \{|a_n|\}_{n=0}^{\infty}$, and $\|Ta\|_{p,s} \leq C(p, s) \|a\|_{p,s}$ by (a). Furthermore, there is $f \in L^{p',s}[0, \infty)$ such that

$$(**) \quad |(Ta, f)| = C(p, s) \|a\|_{p,s} \|f\|_{p',s}.$$

Since $|(Ta, f)| \leq (Ta, |f|)$ for $a_n \geq 0$, and $|(Ta, f)| \leq C(p, s)\|a\|_{p,s}\|f\|_{p',s}$ by (a) it follows also that $f \geq 0$. Thus $(**)$ and, therefore, a_n is a null sequence by the remark made at the end of the proof for (a).

(ii) We now proceed to establish that $\|T\| = C(p, s)$ or, what is equivalent, $C(p, s)$ is best possible constant for the inequality, $\|Ta\|_{p,s} \leq C(p, s)\|a\|_{p,s}$ for $1 \leq p \leq \infty$. Since $\|Ta\|_{p,s} = C(p, s)\|a\|_{p,s}$ for $p = 1$, $a_n \geq 0$, and, for $p = \infty$ when $a_n \geq 0$ is a constant, we may assume $1 < p < \infty$. For this it suffices to show that for each $0 < \eta < 1$ there exists a sequence a_n in $l^{p,s}$ such that

$$\|Ta\|_{p,s}^p > (1 - \eta)^{2p+1} C(p, s)^p \|a\|_{p,s}^p.$$

Let $a_n = (n + s + 1)^{-1/p - \varepsilon} (n + 1)_s^{-1/p}$, for $\varepsilon > 0$, $s > -1$. Then

$$\|a\|_{p,s}^p = \sum_{n=0}^{\infty} (n + s + 1)^{-1 - \varepsilon p},$$

and from the estimates for $(n + 1)_s$ in Lemma 1 we obtain

$$\begin{aligned} a_n &\geq (n + s + 1)^{-\varepsilon} (n + 1)^{-(1+s)/p} \geq (n + 1)^{-(1+s)/p - \varepsilon} \quad \text{for } -1 < s < 0, \\ a_n &\geq (n + s + 1)^{-(1+s)/p - \varepsilon} \quad \text{for } s \geq 0. \end{aligned}$$

Therefore, with $s^* = \max(s, 0)$, $\gamma = 1/\Gamma((1 + s)/p + \varepsilon)$ for all $\varepsilon > 0$, $s > -1$ and positive integer $N \geq 2$, we have

$$\begin{aligned} a_n &\geq (n + s^* + 1)^{-(1+s)/p - \varepsilon} = \gamma \int_0^{\infty} x^{(1+s)/p + \varepsilon - 1} e^{-(n + s^* + 1)x} dx, \\ (Ta)(y) &\geq \gamma \int_0^{\infty} d\sigma(t) \int_0^{\infty} x^{(1+s)/p + \varepsilon - 1} \left[\sum_{n=0}^{\infty} e^{-yt} \frac{(yt)^n}{n!} e^{-(n + s^* + 1)x} \right] dx \\ &= \gamma \int_0^{\infty} d\sigma(t) \int_0^{\infty} x^{(1+s)/p + \varepsilon - 1} e^{-(s^* + 1)x} e^{yt(e^{-x} - 1)} dx \\ &> \gamma \int_0^{\infty} d\sigma(t) \int_0^{\infty} x^{(1+s)/p + \varepsilon - 1} e^{-(1 + s^* + yt)x} dx \\ &\quad \text{(since } e^{-x} - 1 > -x \text{ for } x > 0) \\ &= \int_0^{\infty} (1 + s^* + yt)^{-(1+s)/p - \varepsilon} d\sigma(t), \\ \|Ta\|_{p,s}^p &> \int_0^{\infty} y^{-(1 + \varepsilon p)} dy \left[\int_0^{\infty} \left(\frac{1 + s^*}{y} + t \right)^{-(1+s)/p - \varepsilon} d\sigma(t) \right]^p \\ &> \int_{(N+s)}^{\infty} y^{-(1 + \varepsilon p)} \left[\int_{1/(N+s)}^{\infty} \left(\frac{1 + s^*}{N + s} + t \right)^{-(1+s)/p - \varepsilon} d\sigma(t) \right]^p \\ &> \sum_{n=N}^{\infty} (n + s + 1)^{-(1 + \varepsilon p)} \left[\int_{1/(N+s)}^{\infty} \left(\frac{1 + s^*}{N + s} + t \right)^{-(1+s)/p - \varepsilon} d\sigma(t) \right]^p. \end{aligned}$$

Now for a given $0 < \eta < 1$, we choose $N \geq 2$, $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$ in that order so that

$$\begin{aligned} & \int_{1/(N+s)}^{\infty} \left(\frac{1+s^*}{N+s} + t \right)^{-(1+s)/p} d\sigma(t) \\ & > (1-\eta) \int_0^{\infty} t^{-(1+s)/p} d\sigma(t) = (1-\eta)C(p, s), \\ & \int_{1/(N+s)}^{\infty} \left(\frac{1+s^*}{N+s} + t \right)^{-(1+s)/p-\varepsilon} d\sigma(t) \\ & > (1-\eta) \int_{1/(N+s)}^{\infty} \left(\frac{1+s^*}{N+s} + t \right)^{-(1+s)/p} d\sigma(t) \quad \text{for } 0 < \varepsilon < \varepsilon_1, \\ & \sum_{n=N}^{\infty} (n+s+1)^{-(1+\varepsilon p)} > (1-\eta) \sum_{n=0}^{\infty} (n+s+1)^{-(1+\varepsilon p)} \quad \text{for } 0 < \varepsilon < \varepsilon_2. \end{aligned}$$

The selection of ε_1 is possible by dominated convergence since

$$\left(\frac{1+s^*}{N+s} + t \right)^{-(1+s)/p-\varepsilon} \leq \left(\frac{N+s^*}{N+s} \right)^{\varepsilon} t^{-(1+s)/p} \quad \text{for } t \geq \frac{1}{N+s}.$$

The selection of ε_2 is also possible in view of the equation

$$\sum_{n=0}^{\infty} (n+s+1)^{-(1+\varepsilon p)} = \sum_{n=0}^{N-1} (n+s+1)^{-(1+\varepsilon p)} + \sum_{n=N}^{\infty} (n+s+1)^{-(1+\varepsilon p)}$$

where the ratio of the second term on the right-hand of the above equation to that on the left-hand side increases to 1 as ε decreases to zero. Thus if ε is chosen smaller than both ε_1 and ε_2 and the estimates are combined we obtain

$$\|Ta\|_{p,s}^p > (1-\eta)^{2p+1} \|a\|_{p,s}^p.$$

(iii) Suppose T is bounded and consider the transformation T_{δ} generated by the restriction of the measure σ on $[\delta, \infty)$ for $\delta > 0$. Clearly, by (a) T_{δ} is bounded and, by (ii) of (b), $\|T_{\delta}\| = \int_{\delta}^{\infty} t^{-(1+s)/p} d\sigma(t) < \infty$. Furthermore, $|(T_{\delta}a)(y)| \leq (T_{\delta}|a|)(y) \leq (T|a|)(y)$ for $y \geq 0$. Hence, $\|T_{\delta}\| \leq \|T\|$ which yields that $C(p, s) \leq \|T\| < \infty$ on letting $\delta \rightarrow 0$. This completes the proof of Theorem 1.

THEOREM 2. Consider the transformation T_{λ} given by

$$(T_{\lambda}a)(y) = y^{1-\lambda} \sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-yt} \frac{(yt)^n}{n!} \psi(t) dt$$

where

$$C(\lambda, p, r) = \int_0^{\infty} t^{-1/\lambda r} |\psi(t)|^{1/\lambda} dt < \infty, \quad 1/r = 1/p + \lambda - 1.$$

Then T_{λ} is a bounded transformation from l^p to $L^r[0, \infty)$ for $1 < p < r < \infty$, and $\|T_{\lambda}a\|_r \leq C(\lambda, p, r)^{\lambda} \|a\|_p$. Moreover, if $\psi \geq 0$, then $\|T_{\lambda}a\|_r < C(\lambda, p, r)^{\lambda} \|a\|_p$ unless a_n is a null sequence.

PROOF. Let $f \in L^{r'}[0, \infty)$, $0 < \lambda < 1$, $1 < p < r < \infty$, and let $(E_t a)(y)$ be the transformation used in the proof of Theorem 1(a). Then an application of Hoelder's

inequality for the indices $1/r + 1/p' + (1 - \lambda) = 1$, yields

$$\begin{aligned} \left| \int_0^\infty (T_\lambda a)(y) f(y) dy \right| &\leq \int_0^\infty \int_0^\infty [(E_t a)(y) t^{1/\lambda p'} |\psi(t)|^{1/\lambda}]^{1/r} \\ &\quad \cdot [|f(y)|^{r'} t^{-1/\lambda r} |\psi(t)|^{1/\lambda}]^{1/p'} [E_t a)(y)^p |f(y)|^{r'} y]^{1-\lambda} dy dt \\ &\leq \left[\int_0^\infty \int_0^\infty (E_t a)(y)^p t^{1/\lambda p'} |\psi(t)|^{1/\lambda} dy dt \right]^{1/r} \\ &\quad \cdot \left[\int_0^\infty \int_0^\infty |f(y)|^{r'} t^{-1/\lambda r} |\psi(t)|^{1/\lambda} dy dt \right]^{1/p'} \\ &\quad \cdot \left[\int_0^\infty \int_0^\infty (E_t a)(y)^p |f(y)|^{r'} y dy dt \right]^{1-\lambda} = P^{1/r} Q^{1/p'} R^{1-\lambda}. \end{aligned}$$

Since $1/\lambda p' + 1/\lambda r = 1$ and

$$(E_t a)(y)^p \leq \sum_{n=0}^{\infty} e^{-yt} \frac{(yt)^n}{n!} |a_n|^p$$

by Hoelder's inequality, we have

$$\begin{aligned} P &\leq \int_0^\infty t^{-1/\lambda r} |\psi(t)|^{1/\lambda} dt \left[\sum_{n=0}^{\infty} |a_n|^p \int_0^\infty e^{-yt} \frac{(yt)^n}{n!} t dy \right] = C(\lambda, p, r) \|a\|_p^p, \\ Q &= C(\lambda, p, r) \|f\|_{r'}^{r'}, \\ R &\leq \int_0^\infty |f(y)|^{r'} dy \left[\sum_{n=0}^{\infty} |a_n|^p \int_0^\infty e^{-yt} \frac{(yt)^n}{n!} y dt \right] = \|f\|_{r'}^{r'} \|a\|_p^p. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \int_0^\infty (T_\lambda a)(y) f(y) dy \right| &\leq C(\lambda, p, r)^{1/r+1/p'} \|a\|_p^{p(1/r+1-\lambda)} \|f\|_{r'}^{r'(1/p'+1-\lambda)} \\ &= C(\lambda, p, r)^\lambda \|a\|_p \|f\|_{r'} \end{aligned}$$

and

$$\|T_\lambda a\|_r \leq C(\lambda, p, r)^\lambda \|a\|_p.$$

The proof for the assertion when $\psi \geq 0$ is similar to that given for Theorem 1.

Corollary 1 is simply the special case of Theorem 1 where the measure σ is the unit mass at 1, and for $s = 0$ it includes an inequality of Knopp for the Borel transform of a sequence [8, Satz III].

COROLLARY 1. For $s > -1$, $1 \leq p \leq \infty$,

$$\int_0^\infty \left| \sum_{n=0}^{\infty} e^{-y} \frac{y^n}{n!} a_n \right|^p y^s dy \leq \sum_{n=0}^{\infty} |a_n|^p (n+1)_s$$

where the constant is best possible, and for $1 < p < \infty$ the strict inequality is valid unless the sequence a_n is null.

The generalized Abel means $A^{(\nu)}(x)$ of order $\nu > 0$ of a sequence a_n and the modified moments $A_n^{(\nu)}$, $\nu > 0$, of a function $a(x)$, $0 \leq x \leq 1$, are defined by

$$A^{(\nu)}(x) = (1-x)^\nu \sum_{n=0}^{\infty} \binom{n+\nu-1}{\nu-1} a_n x^n,$$

$$A_n^{(\nu)} = \binom{n+\nu-1}{\nu-1} \int_0^1 x^n a(x) dx,$$

for $0 \leq x \leq 1$, and $n \geq 0$, respectively. Note that for $\nu = 1$ these are simply the ordinary Abel means and moments. In the sequel we also use the notation $A(x)$ for the power series $A^{(1)}(x)/(1-x)$ with coefficients a_n and A_n for the moments $A_n^{(1)}$ of the function $a(x)$.

COROLLARY 2. For $1 < p < \infty$, $\nu > 0$, and $-1 < s < \nu p - 1$,

$$(a) \quad \int_0^1 |A^{(\nu)}(x)|^p (1-x)^{-s-2} x^s dx < \left[\frac{\Gamma((\nu p - s - 1)/p)}{\Gamma(\nu)} \right]^p \sum_{n=0}^{\infty} |a_n|^p (n+1)_s,$$

$$(b) \quad \sum_{n=0}^{\infty} |A_n^{(\nu)}|^p (n+1)_s^{1-p} < \left[\frac{\Gamma((\nu p' - s - 1)/p')}{\Gamma(\nu)} \right]^p \int_0^1 |a(x)|^p (1-x)^{(2-\nu)p-s-2} x^s dx$$

unless the sequence a_n is null or $a(x)$ is null a.e., and the constants are best possible.

PROOF. (a) follows from Theorem 1 on putting the gamma probability density with parameter $\nu > 0$ for the measure σ . Then

$$C(p, s) = \frac{\Gamma((\nu p - s - 1)/p)}{\Gamma(\nu)} \quad \text{for } -1 < s < \nu p - 1,$$

$$(Ta)(y) = \sum_{n=0}^{\infty} a_n \int_0^{\infty} \frac{e^{-yt} (yt)^n}{n!} t^{\nu-1} e^{-t} dt$$

$$= (1+y)^{-\nu} \sum_{n=0}^{\infty} \binom{n+\nu-1}{\nu-1} a_n \left(\frac{y}{1+y} \right)^n,$$

and, the change of variable $x = y/(1+y)$, which maps $[0, \infty)$ onto $[0, 1)$ so that $(Ta)(y) = A^{(\nu)}(x)$ and $y^s dy = (1-x)^{-s-2} x^s dx$, shows that (a) is simply a re-statement of (a) and (b) of Theorem 1. (b) Let σ be a finite positive measure such that $C(p', s) < \infty$. Then it is easy to see that the transformation T^* given for $f \in L^{p,s}[0, \infty)$ by

$$(T^* f)_n = \int_0^{\infty} f(y) y^s dy \int_0^{\infty} \frac{e^{-yt} (yt)^n}{\Gamma(n+s+1)} d\sigma(t)$$

is the adjoint of the transformation T in Theorem 1. Hence, T^* is a bounded transformation from $L^{p,s}[0, \infty)$ into $l^{p,s}$, $\|T^*\| = \|T\| = C(p', s)$ and $\|T^* f\|_{p,s} < C(p', s) \|f\|_{p,s}$ unless f is null a.e. Let σ be the gamma probability density as before in (a). Then

$$C(p', s) = \Gamma((\nu p' - s - 1)/p') / \Gamma(\nu),$$

and

$$(T^*f)_n = (n+1)_s^{-1} \binom{n+\nu-1}{\nu-1} \int_0^\infty \left(\frac{y}{1+y} \right)^n (1+y)^{-\nu} f(y) y^s dy.$$

Now let $A_n^{(\nu)}$ be the modified moments of the function $a(x)$, $0 \leq x \leq 1$. Then, through the change of variable $y = x/(1-x)$ which maps $[0, 1]$ onto $[0, \infty)$ so that $dx = dy/(1+y)^2$, and $y^s dy = (1-x)^{-s-2} x^s dx$, we see that $A_n^{(\nu)}/(n+1)_s = (T^*f)_n$ where $f(y) = (1+y)^{\nu-2} a(y/(1+y))$, and conclude that the assertion made earlier concerning the transformation T^* translates precisely into that of (b) which we wanted to show.

COROLLARY 3. *If $1 < p < r < \infty$, $1/r = 1/p + \lambda - 1$, $\nu > 1/p$, then*

$$\left[\int_0^1 |A^{(\nu)}(x)|^r (1-x)^{-r/p-1} x^{r/p-1} dx \right]^{1/r} < \left[\frac{\lambda^{(\nu p-1)/\lambda p} \Gamma((\nu p-1)/\lambda p)}{\Gamma(\nu)} \right]^\lambda \left[\sum_{n=0}^\infty |a_n|^p \right]^{1/p}$$

unless the sequence a_n is null.

The proof for this is identical with that given for Corollary 2(a) except that here Theorem 2 is used in place of Theorem 1.

REMARK. Theorems 1, 2, and Corollaries 2(a), 3 remain valid when $\|a\|_p$ is replaced by $\|Ba\|_p$ where Ba is the Borel transform of the sequence a_n . The proofs are similar except for the assertions about the constants being best possible, which follow by Corollary 1 from Theorem 1(b) and Corollary 2(a).

Hardy and Littlewood in [4] developed certain weighted normed inequalities, which relate the norm of a power series $A(x)$ to that of its coefficients a_n , and the norm of a moment sequence A_n to that of its corresponding function $a(x)$, under the assumption that $a_0 = 0$ and with the exclusion of the term A_0 from their estimates. We now want to show how the inequalities obtained for the generalized Abel means $A^{(\nu)}(x)$ and the modified moments $A_n^{(\nu)}$ lead to improved forms of their inequalities so that no restriction is made on a_0 and the term A_0 is also included. Theorem 3 corresponds to Theorems 3 and 4 of Hardy and Littlewood which are the most inclusive results for $p > 1$ in [4]. The inequalities listed below constitute a considerable improvement of the inequalities (A_0) , (B_0) , (C_0) and (D_0) in [4]. (A_0) and (D_0) are simply special cases of Corollary 2 where $\nu = 1$, $s = 0$. Similarly, (B_0) and (C_0) are special cases of Theorem 3 where $r = p$, $q = p$ in (a), and $r = p$, $q = p'$ in (b). Note that the constants in (B_0) for $2 \leq p < \infty$ and in (C_0) for $1 < p \leq 2$ are, respectively, $\Gamma(1/p)$ and $\Gamma(1/p')$. But (A_0) or, what is equivalent, Corollary 1(a) is false for $p = 1$. Because Theorem 1 is false in the case of the ordinary Abel means, i.e., when $\nu = 1$, $p = 1$, $s = 0$, and the constant $C(p, s)$ is infinite. (C_0) is also false for $p = 1$, since $A_n = 1/\log(n+2)$, $n \geq 0$, is a moment sequence and the series in (C_0) is divergent [3, §11.13]. However, (A_0^*) and (C_0^*) , which are analogues of Carleman's inequality, serve as substitutes for (A_0) and (C_0) when $p = 1$, $a(x) \geq 0$, $a_n \geq 0$, and where γ is Euler's constant [5, p. 249]. They are derived by replacing

a_n by $a_n^{1/p}$ in (A_0) , $a(x)$ by $a(x)^{1/p}$ in (C_0) and then taking the limit as $p \rightarrow \infty$.

$$(A_0) \quad \int_0^1 |A(x)|^p (1-x)^{p-2} dx < \Gamma\left(\frac{1}{p'}\right)^p \sum_{n=0}^{\infty} |a_n|^p,$$

$$(B_0) \quad \int_0^1 |A(x)|^p dx < \left[\Gamma\left(\frac{1}{p}\right) \max\left(\frac{1}{\Gamma(2/p)}, 1\right) \right]^p \sum_{n=0}^{\infty} |a_n|^p (n+1)^{p-2},$$

$$(C_0) \quad \sum_{n=0}^{\infty} |A_n|^p (n+1)^{p-2} < \left[\Gamma\left(\frac{1}{p'}\right) \max\left(\frac{1}{\Gamma(2/p')}, 1\right) \right]^p \int_0^1 |a(x)|^p dx,$$

$$(D_0) \quad \sum_{n=0}^{\infty} |A_n|^p < \Gamma\left(\frac{1}{p}\right)^p \int_0^1 |a(x)|^p (1-x)^{p-2} dx,$$

unless the sequence a_n is null or $a(x)$ is null a.e., respectively, for $1 < p < \infty$, and the constants are best possible in (A_0) and (D_0) for $1 < p < \infty$, in (B_0) for $2 \leq p < \infty$, and in (C_0) for $1 < p \leq 2$.

$$(A_0^*) \quad \int_0^1 (1-x)^{-2} \exp\left[(1-x) \sum_{n=0}^{\infty} (\log a_n) x^n\right] dx$$

$$\leq e^{\gamma} \sum_{n=0}^{\infty} a_n, \quad a_n \geq 0,$$

$$(C_0^*) \quad \sum_{n=0}^{\infty} (n+1)^{-2} \exp\left[(n+1) \int_0^1 x^n \log a(x) dx\right]$$

$$\leq e^{2-\gamma} \int_0^1 a(x) dx, \quad a(x) \geq 0.$$

For $\nu = 1$, it follows from Corollary 2, where $s = p/q' - 1$ in (a) and $s = p'/q' - 1$ in (b), and Lemma 1(a), that Theorem 3 is also valid when $r = p$, with $1 < q \leq p'$ in (a) and $1 < q \leq p$ in (b), provided that $|a_0|^p$ is replaced by $|a_0|^p \Gamma(p/q')$, $|A_0|^p$ by $|A_0|^p \Gamma(p'/q')$, and the weight $(n+1)$ by n for $n \geq 1$. Moreover, the resulting constant $\Gamma(1/q)^p$ is best possible by the same argument given below for (a) when $r = p$, $q \geq p'$. This applies in particular to (B_0) where $q = p \leq p'$ for $1 < p \leq 2$ and to (C_0) where $q = p' \leq p$ for $p \geq 2$.

THEOREM 3. For $1 < p \leq r < \infty$, $q > 0$, we have

$$(a) \quad \left[\int_0^1 |A(x)|^r (1-x)^{r/q-1} dx \right]^{1/r} < K(p, r, q) \left[\sum_{n=0}^{\infty} |a_n|^p (n+1)^{-(p+q-pq)/q} \right]^{1/p},$$

$$(b) \quad \left[\sum_{n=0}^{\infty} |A_n|^r (n+1)^{r/q-1} \right]^{1/r} < K(r', p', q) \left[\int_0^1 |a(x)|^p (1-x)^{-(p+q-pq)/q} dx \right]^{1/p}$$

unless the sequence a_n is null or $a(x)$ is null a.e., respectively, and where

$$K(p, r, q)^r = [\Gamma(1/q) \max(1/\Gamma(1/p + 1/q), 1)]^p \max(\Gamma(p'/q), 1)^{(r-p)/p'}.$$

Moreover, the constant in (a) is best possible for $r = p$, $q \geq p'$, and in (b) for $r = p$, $q \geq p$.

PROOF. For $p = r$, (a) follows from Corollary 2(a) for $\nu = 1/p + 1/q$, $s = 0$, when applied to the sequence $|a_n|(n+1)^{1-\nu}$, and then the lower estimates for the binomial coefficients in Lemma 2 are substituted in the resulting expression for $A^{(\nu)}(x)$. For $p = r$,

$$K(p, p, q) = \Gamma(1/q) \max(1/\Gamma(1/p + 1/q), 1).$$

For the general case where $r > p$, with S_p denoting the normed sum on the right-hand side of (a), $c = p'/q$, and $\nu = 1/p + 1/q$, we have

$$\begin{aligned} |A(x)| &\leq \sum_{n=0}^{\infty} (n+1)^{1-\nu} (|a_n|(n+1)^{\nu-1} x^n) \\ &\leq S_p \left[\sum_{n=0}^{\infty} (n+1)^{c-1} x^n \right]^{1/p'} \\ &\leq S_p \text{Max}(\Gamma(c), 1)^{1/p'} \left[\sum_{n=0}^{\infty} \binom{n+c-1}{c-1} x^n \right]^{1/p'} \\ &= \text{Max}(\Gamma(p'/q), 1)^{1/p'} (1-x)^{-1/q} S_p, \end{aligned}$$

by Hoelder's inequality and Lemma 2. Finally, using (a) for $r = p$ and the preceding estimate for $A(x)$, we obtain

$$\begin{aligned} \int_0^1 |A(x)|^r (1-x)^{r/q-1} dx &\leq \text{Max}(\Gamma(p'/q), 1)^{(r-p)/p'} S_p^{r-p} \int_0^1 |A(x)|^p (1-x)^{p/q-1} dx \\ &< K(p, p, q)^p \text{Max}(\Gamma(p'/q), 1)^{(r-p)/p'} S_p^r = K(p, r, q)^r S_p^r, \end{aligned}$$

which proves (a). (b) is derived from (a) by the same duality argument used by Hardy and Littlewood to prove Theorem 4 from Theorem 3 in [4]. In fact, a similar duality argument obtains (a) from (b). Consequently, it suffices to show that the constant is best possible only in (a) when $r = p$ and $q \geq p'$ in which case the constant is $\Gamma(1/q)$. For $0 < \varepsilon < 1/q < 1$, define the sequence a_n so that $a_0 = \Gamma(1/q - \varepsilon)$, $a_n = n^{\nu-1}$ for $n \geq 1$, where $\nu = 1/q - \varepsilon$. Then we have

$$\begin{aligned} S_p^p - \Gamma(\nu)^p &= \sum_{n=1}^{\infty} n^{p(\nu-1)} (n+1)^{-(p+q-pq)/q} = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{p(1-\nu)} (n+1)^{-(1+\varepsilon p)} \\ &= \sum_{n=1}^{\infty} (n+1)^{-(1+\varepsilon p)} + \sum_{n=1}^{\infty} \left[\left(1 + \frac{1}{n}\right)^{p(1-\nu)} - 1 \right] (n+1)^{-(1+\varepsilon p)} \\ &\leq \frac{1}{\varepsilon p} + p(1-\nu) \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{p(1-\nu)-1} n^{-(2+\varepsilon p)} \\ &= \frac{1}{\varepsilon p} + O(1) \end{aligned}$$

as $\varepsilon \rightarrow 0$, since $p(1 - \nu) - 1 > 0$ for $q \geq p'$. Therefore, $S_p^p \leq 1/\varepsilon p + O(1)$ where $O(1) > 0$, and $1/\varepsilon p \geq S_p^p/(1 + o(1))$ as $\varepsilon \rightarrow 0$ where $o(1) > 0$. Finally from Lemma 2(c) and the last estimate it follows that

$$\begin{aligned} A(x) &= \Gamma\left(\frac{1}{q-\varepsilon}\right) + \sum_{n=1}^{\infty} n^{\nu-1} x^n \\ &\geq \Gamma\left(\frac{1}{q-\varepsilon}\right) \sum_{n=0}^{\infty} \binom{n+\nu-1}{\nu-1} x^n = \Gamma\left(\frac{1}{q-\varepsilon}\right) (1-x)^{-1/q-\varepsilon}, \\ \int_0^1 |A(x)|^p (1-x)^{p/q-1} dx &> \frac{\Gamma(1/q)}{\varepsilon p} \geq \frac{\Gamma(1/q)^p S_p^p}{1+o(1)}, \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

This shows that the constant is best possible and completes the proof.

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DEPARTMENT OF MATHEMATICS, DEPAUL UNIVERSITY, CHICAGO, ILLINOIS 60614